The spherical MHD code MagIC

Advanced (2/2)

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Outline

- MagIC simulates rotating fluid dynamics in a spherical shell
- It solves for the coupled evolution of Navier-Stokes equation, MHD equation, temperature (or entropy) equation and an equation for chemical composition under both the anelastic and the Boussinesq approximations
- A dimensionless formulation of the equations is assumed
- MagIC is a free software (GPL), written in Fortran
- Post-processing relies on python libraries
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- MagIC uses spherical harmonic decomposition in the angular directions
- Chebyshev polynomials or finite differences are employed in the radial direction
- **MagIC uses a mixed implicit/explicit time stepping scheme**
- The code relies on a hybrid parallelisation scheme (MPI/OpenMP)

Spectral poloidal dynamo equation

Equation for each spherical harmonic degree and order

$$\frac{\ell(\ell+1)}{r^2}\left[\left(\frac{\partial}{\partial t}+\frac{1}{Pm}\frac{\ell(\ell+1)}{r^2}\right)\mathcal{C}_n-\frac{1}{Pm}\mathcal{C}_n''\right]g_{\ell n}^m=\mathcal{N}_\ell^m$$

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How to deal with the time integration, i.e. discretisation in time?

Strategy

- Strong non linearities = **stiff problem**. Usually, higher-order schemes (RK4), or multi-step algorithms are employed (BDF).

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Adopted strategy

Most authors adopt a mixed implicit/explicit algorithm

Semi-implicit scheme

Generic evolution equation with terms $\mathcal{I}(x, t)$ to be treated **implicitly** and $\mathcal{E}(x, t)$ to be treated explicitly:

$$\frac{\partial x}{\partial t} + \mathcal{I}(x, t) = \mathcal{E}(x, t)$$

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Glatzmaier's (1984) time integration scheme (2nd order): Implicit **Crank-Nicolson** scheme:

$$\left(\frac{x(t+\delta t) - x(t)}{\delta t}\right)_{\mathcal{I}} = -\alpha\,\mathcal{I}(x, t+\delta t) - (1-\alpha)\,\mathcal{I}(x,t)$$

Explicit **2nd order Adams-Bashforth** scheme:

$$\left(\frac{x(t+\delta t) - x(t)}{\delta t}\right)_{\mathcal{E}} = \frac{3}{2}\,\mathcal{E}(x,t) - \frac{1}{2}\,\mathcal{E}(x, t-\delta t)$$

N.B. Other schemes are used in some pseudo-spectral codes: BDF/AB or predictor-corrector

Time stepping scheme

$$\frac{x(t+\delta t)}{\delta t} + \alpha\mathcal{I}(x, t+\delta t) = \frac{x(t)}{\delta t} - (1-\alpha)\mathcal{I}(x,t) + \frac{3}{2}\mathcal{E}(x,t) - \frac{1}{2}\mathcal{E}(x, t-\delta t)$$

- When $\alpha = 0.5$, this is pure CN/AB2 implicit/explicit 2nd order scheme
- Glatzmaier (1984) reported an improved stability when $\alpha = 0.6$ (see MagIC's input namelist)

Treatment of Coriolis force

As an example, Coriolis force that enters the W equation:

$$2\tilde{\rho}\,\mathbf{e_r}\cdot(\mathbf{u}\times\mathbf{e_z}) = 2\sin\theta\,\tilde{\rho}u_\phi = \frac{2}{r}\left(\frac{\partial^2 W}{\partial r\partial\phi} - \sin\theta\frac{\partial Z}{\partial\theta}\right)$$

This yields:

$$\mathrm{Cor}_{\ell n}^m = \frac{2}{r}\left[im\,\mathcal{C}_n'\,W_{\ell n}^m - (\ell-1)c_\ell^m\mathcal{C}_n\,Z_{\ell-1,n}^m + (\ell+2)c_{\ell+1}^m\mathcal{C}_n\,Z_{\ell+1,n}^m\right]$$

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Implicit treatment?

- (ℓ, m) mode coupled with $(\ell+1, m)$ and $(\ell-1, m)$ modes
- Poloidal and toroidal equations coupled
- **Implicit treatment of Coriolis force** = much larger matrix
- In MagIC, Coriolis force is **treated explicitly**...

Poloidal magnetic field time stepping

Again equation for poloidal magnetic field:

$$\frac{\ell(\ell+1)}{r^2}\left[\left(\frac{\partial}{\partial t}+\frac{1}{Pm}\frac{\ell(\ell+1)}{r^2}\right)\mathcal{C}_n-\frac{1}{Pm}\mathcal{C}_n''\right]g_{\ell n}^m=\mathcal{N}_\ell^m$$

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Using the CN/AB2 scheme yields the following linear problem

$$\boxed{\begin{aligned}\left[A_{kn}+\alpha G_{kn}\right]g_{\ell n}^m(t+\delta t)=&\left[A_{kn}-(1-\alpha)G_{kn}\right]g_{\ell n}^m(t)\\ &+\frac{3}{2}D_{kn}(t)-\frac{1}{2}D_{kn}(t-\delta t)\end{aligned}}$$

with

$$A_{kn}=\frac{\ell(\ell+1)}{r_k^2}\frac{\mathcal{C}_n(r_k)}{\delta t};\quad G_{kn}=\frac{\ell(\ell+1)}{r_k^2}\frac{1}{Pm}\left[\frac{\ell(\ell+1)}{r_k^2}\mathcal{C}_n(r_k)-\mathcal{C}''_n(r_k)\right];$$

$$D_{kn}=\mathcal{N}_\ell^m(r_k)$$

Some comments on the time-stepping

- $\mathcal{C}_n(r_k)$, $\mathcal{C}'_n(r_k)$, $\mathcal{C}''_n(r_k)$ are **full matrices**: costly **LU factorisations** required ($\mathcal{O}(N_r^2)$) and possibly large memory imprints
- BUT as long as δt does not change, the left hand-side operator does not change
- **Finite differences in radius yield sparse matrices**: less memory, faster solve (at the price of reduced accuracy though)...

Courant condition

- Explicit treatment of Coriolis force: $\delta t \leq 0.1\, E$
- δt should be **smaller than the advection between two grid points**:

$$\delta t_r \leq \min\left[\frac{\delta r}{|u_r|}\right]\ ;\ \delta t_H \leq \min\left[\left(\frac{r^2}{\ell_{\max}(\ell_{\max}+1)(u_\theta^2+u_\phi^2)}\right)^{1/2}\right]$$

Hence

$$\delta t = C\ \min(\delta t_r, \delta t_H)$$

- In presence of a magnetic field, **another condition on the Alvén velocity** is required

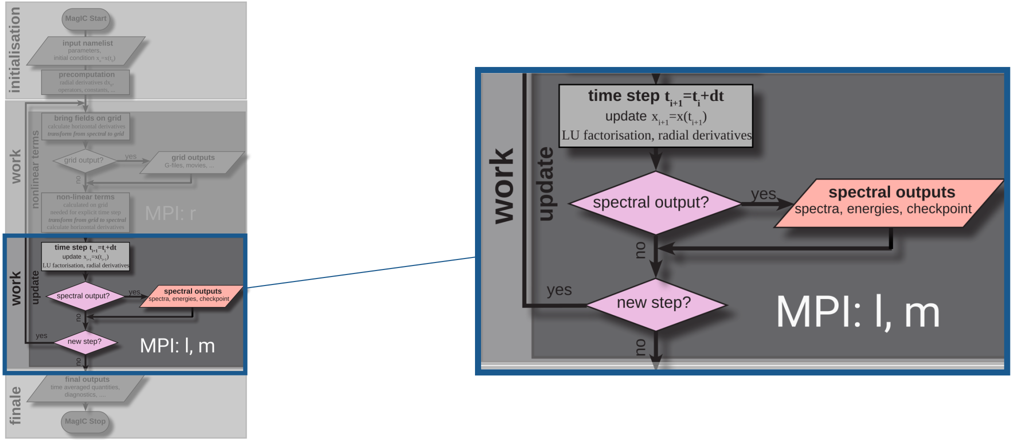
Time integration: summary

Take-away messages on time stepping

- Most of the pseudo-spectral codes assume a **mixed implicit/explicit scheme** (most of the time 2nd order)
- At each time step a linear system needs to be solved
- For Chebyshev-based code: **LU factorisations** $\rightarrow \mathcal{O}(\ell_{\max}^2 N_r^2)$ (matrix can be saved as long as δt does not change though)
- Finite difference are cheaper here: sparse matrix, less memory, faster inversion

▸ Documentation

MagIC structure



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Hybrid configuration used in MagIC

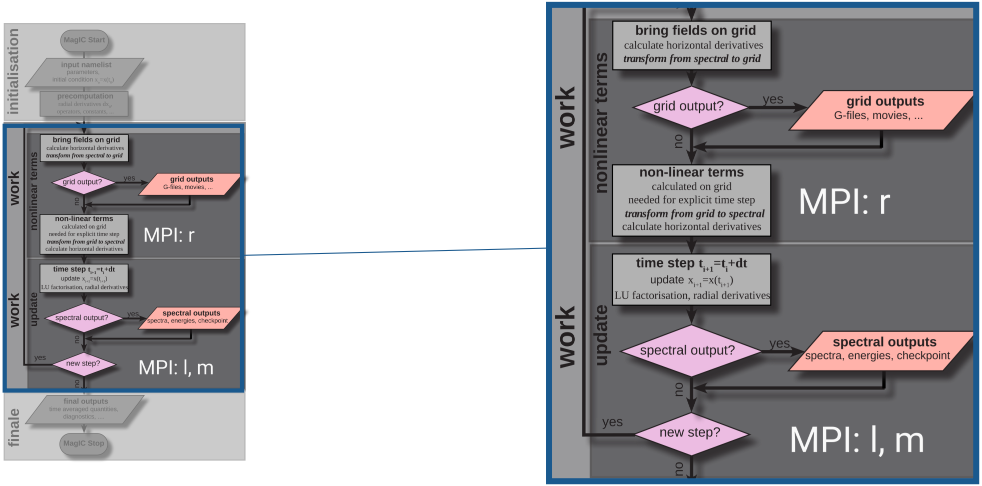
MPI:

- **1st part of the code**: calculation of the nonlinear terms and SH transforms = **radial levels can be treated independently**: r is distributed over N_p MPI ranks
- **2nd part of the code**: time advance of the equations = linear solve = **all the** (ℓ, m) **modes can be treated independently**: (ℓ, m) is distributed over N_p MPI ranks (pairing needed to ensure the load balancing
- **In between**: costly `mpi_all_to_all(...)` calls are required. For large truncations, this becomes a bottleneck...

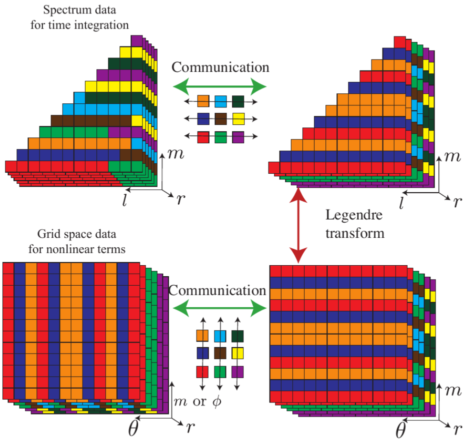
OpenMP:

- **1st part of the code**: N_t `OpenMP` threads can be used over the θ blocks for the SH transforms and computation of nonlinear terms
- **1st part of the code**: N_t `OpenMP` tasks are used over (ℓ, m)

MagIC structure



Possible improvements: 2D-MPI configuration



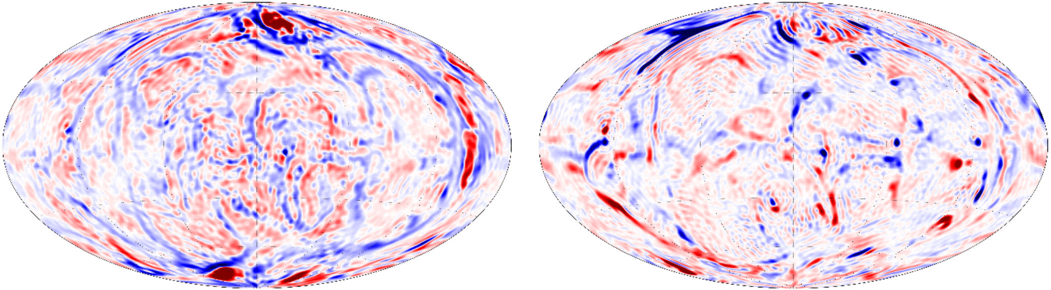
Taken from Calypso's documentation

Outline

How do I know a simulation is under-resolved? (1/2)

- **Look at the solutions!** Usually: flows and magnetic field close to the surfaces are usually prone to under-resolution (boundary layers)

$u_r,\ r = 0.99\, r_o$ $\quad$ $B_r,\ r = 0.99\, r_o$

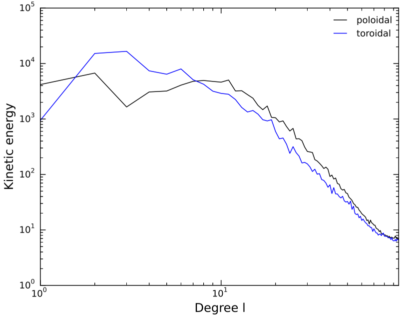


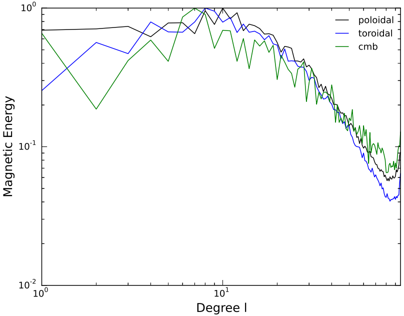
$E = 3 \times 10^{-4},\ Ra = 3 \times 10^6$

- **Obvious signatures of under-resolution**: small-scale structures of comparable size than the grid, "eyes", aliases (sudden localized changes of polarities)

How do I know a simulation is under-resolved? (2/2)

- Look at **spectra** and check the dissipation:

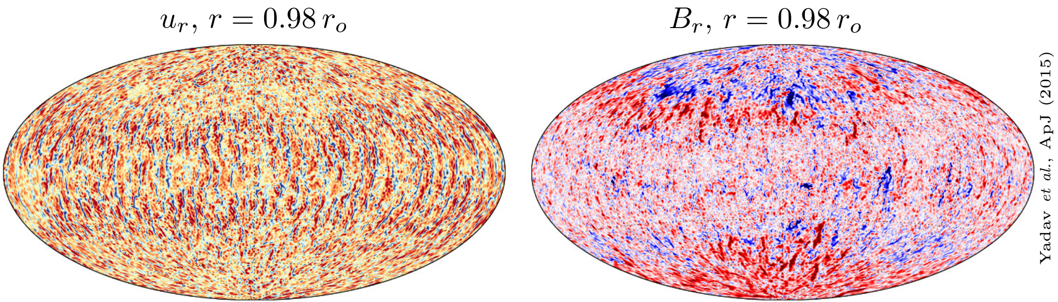




- **Rule of thumb**: 2 orders of magnitude between the **injection scale** and the **dissipation scale**
- **Additional diagnostics** of under-resolution: heat flux conservation, power budget

Possible impacts of under-resolution (1/2)

- Under-resolution might be an issue: it really depends what you are looking at...
- Let's take another example of under-resolution



$N_r = 100,\ N_\theta = 320,\ N_\phi = 640$

- A lot of localised "eyes"

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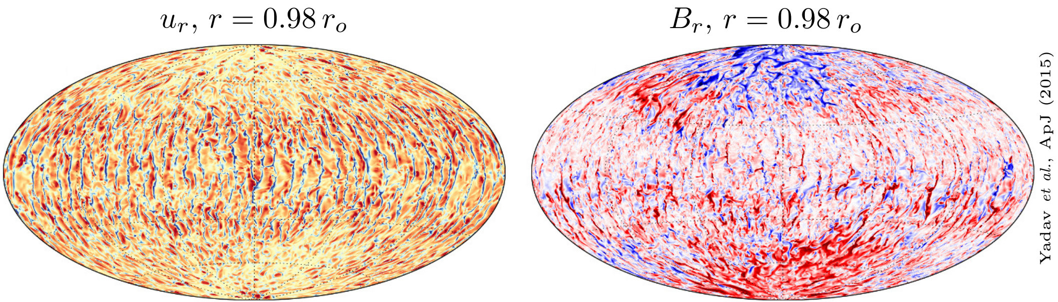
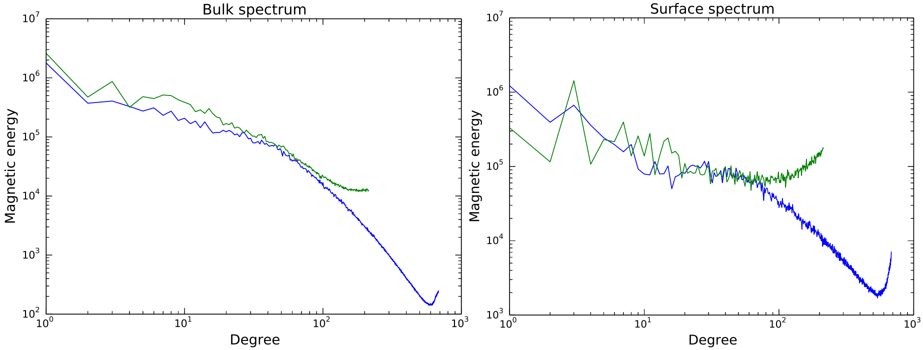


$N_r = 160,\ N_\theta = 1024,\ N_\phi = 2048$

- A lot of localised "eyes"
- **Solution**: multiply the angular resolution by 3

Possible impacts of under-resolution (2/2)



- At first glance, you would better trash the under-resolved case
- **But, the largest scales contributions are reasonably captured**
- Surprisingly, some **global quantities** might still be OK!

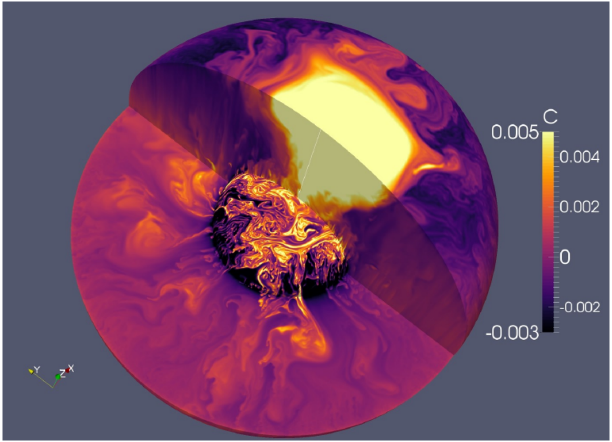
Is it that bad?

Parameter	Under-resolved	Resolved
Volume-averaged quantities		
Rm	1000	1000
Λ	16	16
Surface-averaged quantities		
$\Lambda(r = r_o)$	**53**	**34**
$Nu(r = r_o)$	**1.55**	**1.3**
$Nu(r = r_i)$	**1.3**	**1.3**

Results

- **Global volume-integrated** quantities are still good!
- **But** surface-averaged and **local quantities** are completely wrong
- Be careful with what you are doing!

How much does it cost?



1024^3-class simulation: 10^7 **CPU hours**

Schaeffer *et al.* (2017)

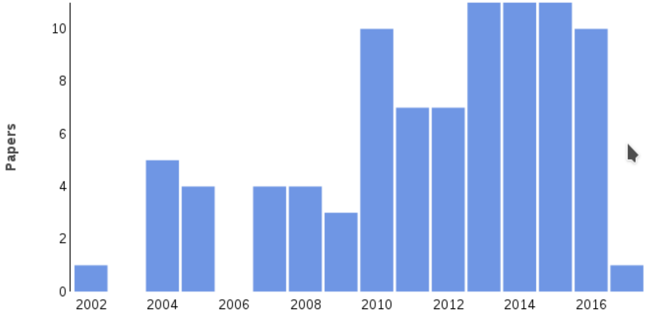
Summary: little recipes for MagIC

1. **Good resolution**: no localised "eyes" or aliases clean spectra
2. **Bad resolution**: aliasing, pile-up of energy, no proper dissipation
3. At some point the simulation will crash (hopefully)...
4. But **some compromises** are possible: slightly under-resolved cases can still provide good **volume-integrated** quantities (numerically cheap)
5. **Be careful though**: local properties (heat transfer, scaling laws) are likely wrong
6. **Don't over-do it**! Large resolution are computationally expensive
7. Why not running first a **smaller and cheaper truncation for transients** and possibly **refining** the grid later on?

Outline

List of publications

To date, around **90 publications** in more than 10 different peer-reviewed journals have been produced using MagIC:

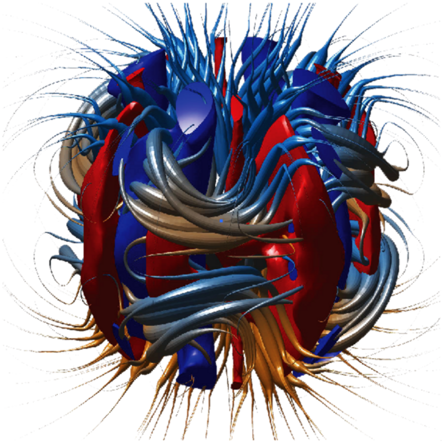


Source ADS Bumblebee

International dynamo benchmark

Christensen *et al.*, PEPI, 2001

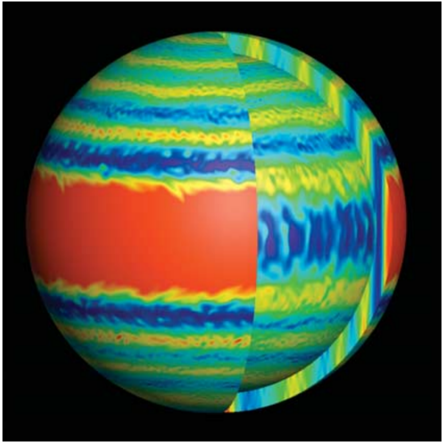
- Earth-like setup
- Boussinesq
- r_i/r_o=0.35
- Weakly-supercritical laminar dynamo
- Code validation



Modelling the Jovian zonal jets

Heimpel *et al.*, Nature, 2005

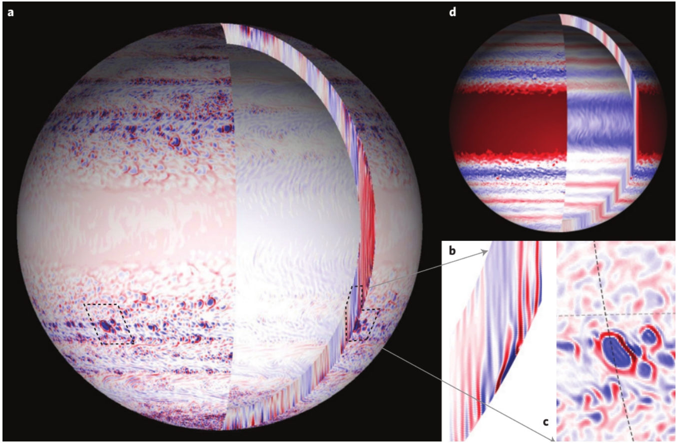
- Jupiter-like zonal jets in a thin convective shell
- Boussinesq
- Non-magnetic
- Stress-free boundaries
- r_i/r_o=0.9
- low E, large Ra



Formation of anti-cyclonic eddies

Heimpel *et al.*, Nat. Geo., 2015

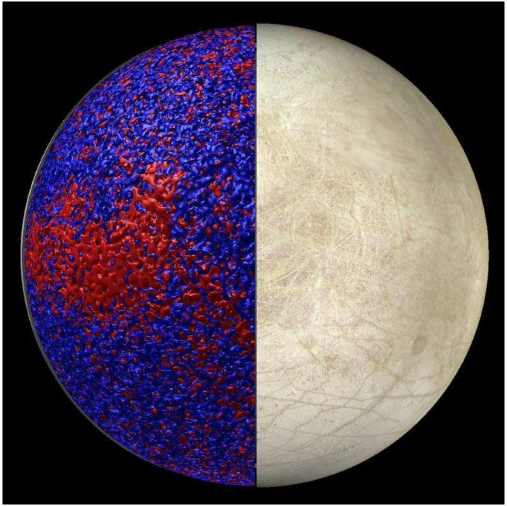
Anelastic, non-magnetic model with a stably-stratified atmosphere.



Explaining chaos terrain on Europa

Soderlund *et al.*, Nat. Geo., 2014

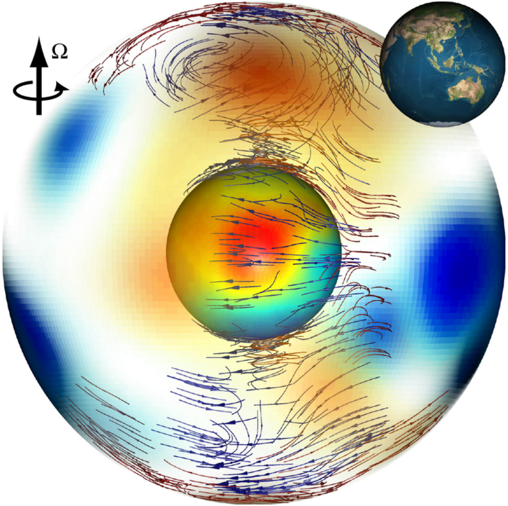
- Europa's ocean
- Thin convective shell
- Boussinesq
- Non-magnetic
- large *Ra*
- stronger equatorial heat flux



Explaining inner core anisotropy

Aubert *et al.*, Nature, 2008

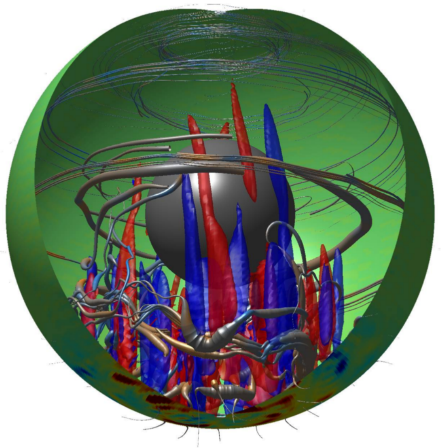
- Geodynamo simulation
- Boussinesq
- Tomographic CMB heat flux pattern
- Inner core anisotropy



Explaining the Martian crustal field anisotropy

Dietrich & Wicht, PEPI, 2013

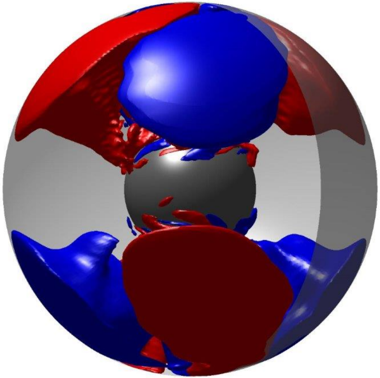
- Increased southern heat flux
- Boussinesq
- Anistropic flow and field



Inertial modes in spherical Couette flows

Wicht, JFM, 2014

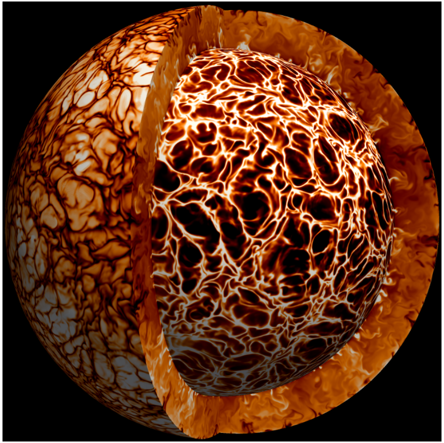
- Spherical Couette
- Boussinesq
- Non-magnetic
- Comparison with Maryland's experiment



Rayleigh-Bénard convection in spherical shells

Gastine *et al.*, JFM, 2015

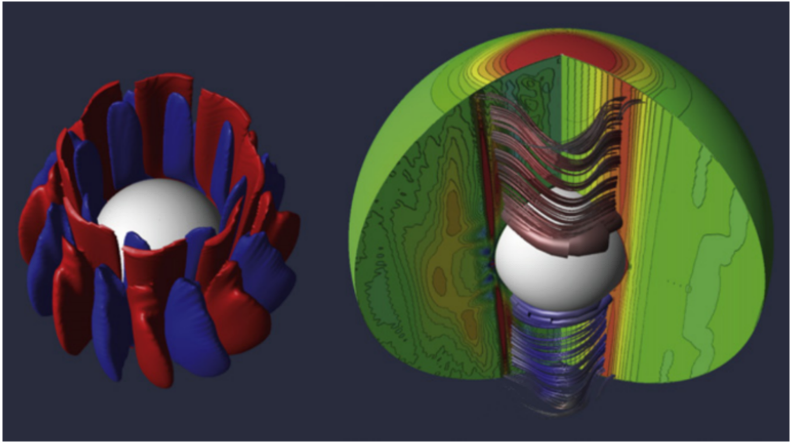
- Non-rotating
- Boussinesq
- Non-magnetic
- high *Ra*



Explaining Saturn's peculiar magnetic field

Cao *et al.*, Icarus, 2012

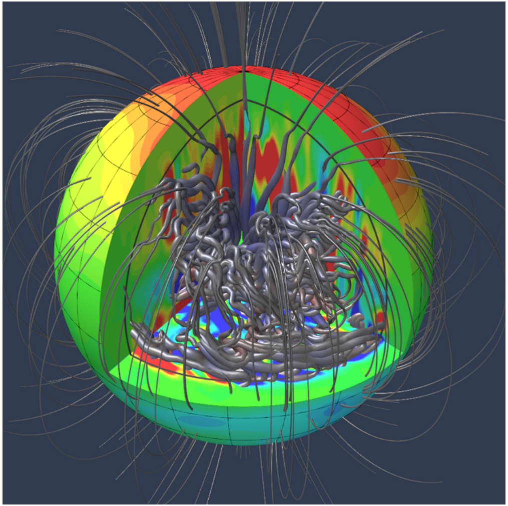
Slightly supercritical spherical Taylor-Couette dynamo



Jupiter hosts two dynamos?

Gastine *et al.*, GRL, 2014

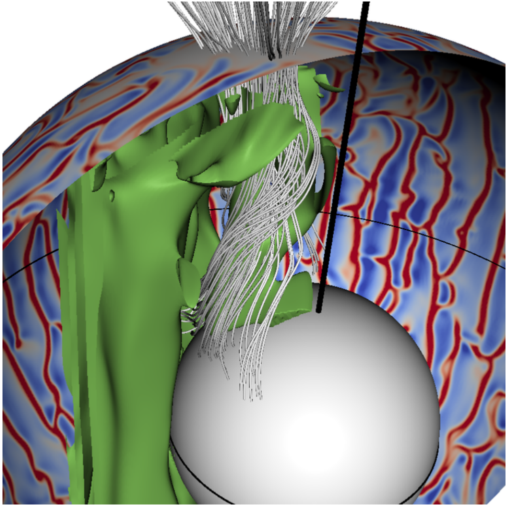
- Jovian-like reference state
- Anelastic
- Magnetic
- low E high Ra



Formation of polar spots on rapidly-rotating cool stars

Yadav *et al.*, ApJ, 2015

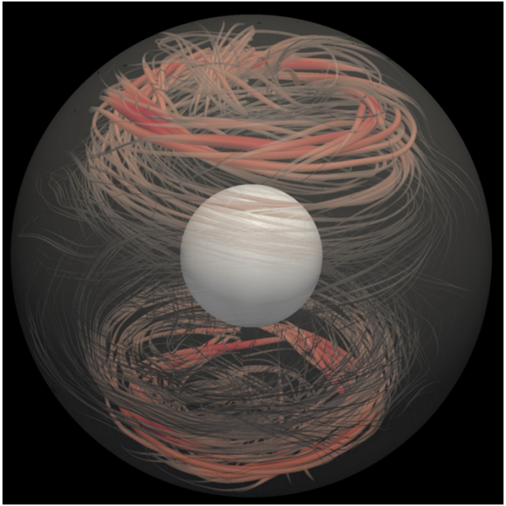
- fully convective M dwarf
- Anelastic
- Magnetic
- Large density contrast



MRI in radiative zones of A-type stars

Jouve *et al.*, A&A, 2015

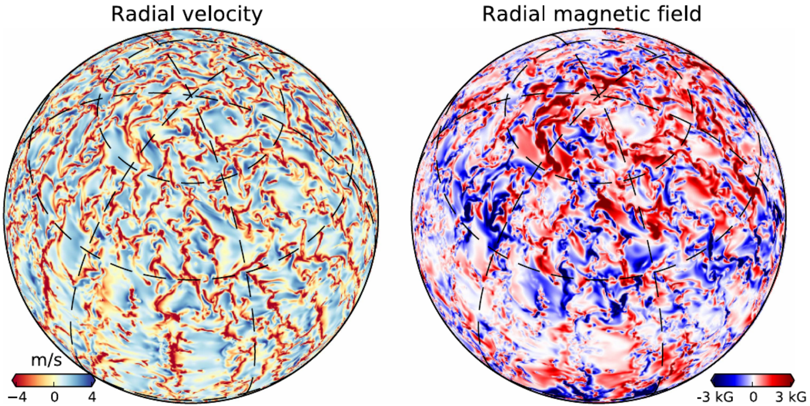
- Incompressible fluid
- Magnetic instabilities (MRI & Tayler)
- Here MRI



Magnetic cycles on Proxima Centauri?

Yadav *et al.*, ApJ, 2016

Fully convective, anelastic, rapidly-rotating dynamo (M dwarf)



What is coming next?

?!

It is up to you now!

Outline

Selected references I

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