Radiation hydrodynamics with Flux Limited Diffusion in RAMSES

Benoît Commerçon

Centre de Recherche Astrophysique de Lyon

CRAL

CENTRE DE RECHERCHE ASTROPHYSIQUE DE LYON

Lecture

- radiation hydrodynamics
  - implicit method
  - flux-limited diffusion approximation

- methods in RAMSES
  - FLD, adaptive-time-stepping
  - interface with RADMC-3D
  - anisotropic diffusion

Exercise

- **Hands-on RAMSES**
  - download the code: https://bitbucket.org/bcommerc/ramses_fld
  - run the test suite (MHD, RHD)
  - play with the namelists

Outline

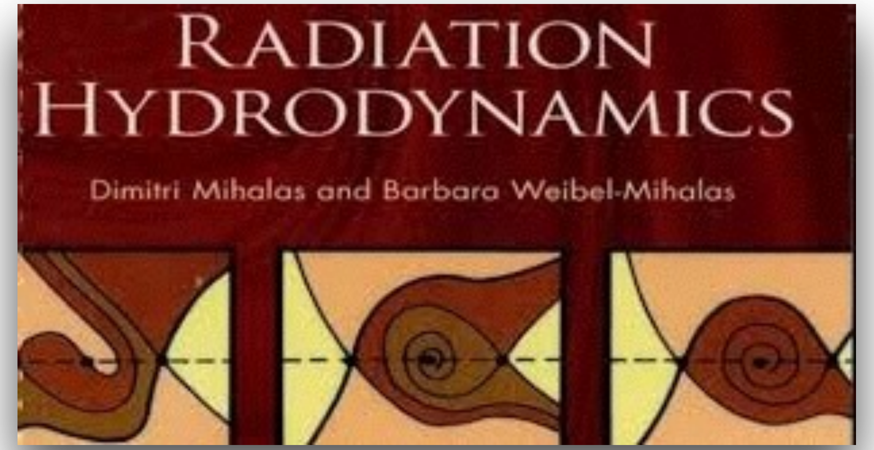
1. **Introduction**
2. **RHD with Grey Flux Limited Diffusion**
    - integration in RAMSES
    - adaptive time-steps
    - tests
3. **Multigroup FLD**
    - scheme
    - tests
4. **Extension to cosmic rays hydrodynamics**

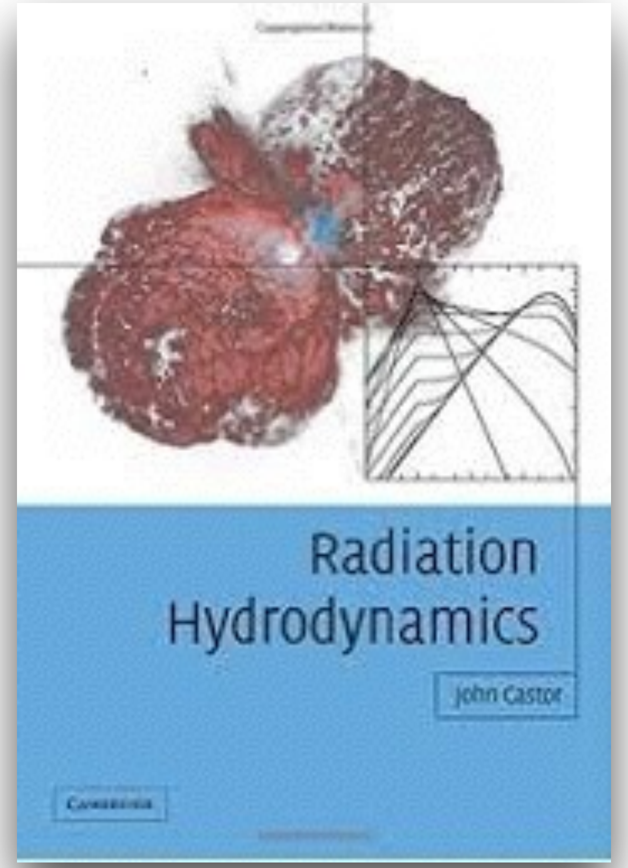
Motivation for radiation hydrodynamics (RHD)

- Add more physics
- Account for radiative transfer feedback on hydrodynamics - combined matter-radiation fluid
    - continuum radiation
    - stellar irradiation
    - ionisation
    - radiation pressure
- Need to known the radiation field intensity in the computational domain

Foundations of radiation-hydrodynamics

- As for MHD, a lot of approximations and sub-grid physics
- Cover a wide range of physical and dynamical scales
- Mathematical problem: hyperbolic-parabolic system
- In which frame should the photons be evaluated? (co-moving, mixed-frame)?
- Frequency dependent transport?
- Non-LTE effects?
- Opacities?





Regimes for radiation transport

Radiation can propagate through the medium in two limiting regimes.

Photon mean free path $\lambda_\nu = \frac{1}{\kappa_\nu}$; Optical depth $\tau_\nu = \kappa_\nu L$

- Free streaming $\lambda_\nu >> L$
- Optically thin $\tau_\nu << 1$

- Diffusion limit $\lambda_\nu << L$
- Optically thick $\tau_\nu >> 1$

If radiation is coupled to the gas, two additional regimes for the diffusion depending on v/c (e.g. *Krumholz et al. 2007*)

- Static diffusion $\tau_\nu \frac{v}{c} << 1$

  e.g., stellar accretion disk

- Dynamic diffusion $\tau_\nu \frac{v}{c} >> 1$

  e.g., stellar interior

Description of radiation field

- Radiation field is a function of position, time, angle & frequency
- Radiation specific intensity $I(\mathbf{x}, t; \mathbf{n}, \nu)$ defined as

$$dE = I(\mathbf{x}, t; \mathbf{n}, \nu) dS \cos(\alpha) d\omega d\nu dt$$

- 7 dimensions => need to reduce dimensionality

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- 7 dimensions => need to reduce dimensionality

- Moments of the specific intensity
  - Energy $$E_\nu(\mathbf{x}, t) = \frac{1}{c} \int I(\mathbf{x}, t; \mathbf{n}, \nu) d\Omega$$
  - Flux $$\mathbf{F}_\nu(\mathbf{x}, t) = \int \mathbf{n} I(\mathbf{x}, t; \mathbf{n}, \nu) d\Omega$$
  - Pressure $$\mathbb{P}_\nu(\mathbf{x}, t) = \frac{1}{c} \int \mathbf{n} \times \mathbf{n} I(\mathbf{x}, t; \mathbf{n}, \nu) d\Omega$$
    $$\mathrm{Tr}(\mathbb{P}_\nu) = E_\nu$$

Description of radiation field

- *Thermal radiation*: assuming Thermodynamical Equilibrium, intensity is described by an isotropic distribution function, the *Planck function*

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT} - 1}$$

- *Stefan law*: integrated energy density for thermal radiation

$$\int_0^\infty B_\nu(T)d\nu = \frac{c}{4\pi}a_\mathrm{r}T^4 = \frac{c}{4\pi}E$$

- One can define a radiation temperature such as

$$T_r = (E/a_r)^{1/4}$$

Radiative transfer equation

- Radiative transfer equation

Emission

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{n}\cdot\nabla\right) I(\mathbf{x},t;\mathbf{n},\nu) = \eta(\mathbf{x},t;\mathbf{n},\nu) - \chi(\mathbf{x},t;\mathbf{n},\nu) I(\mathbf{x},t;\mathbf{n},\nu)$$

Specific intensity

Absorption

- Assuming TE (and neglecting scattering), thermal emission/absorption terms are

$$\eta_{\rm th}(\mathbf{x},t;\mathbf{n},\nu) = \kappa(\mathbf{x},t;\mathbf{n},\nu) B(\mathbf{x},t;\mathbf{n},\nu)$$

$$\chi(\mathbf{x},t;\mathbf{n},\nu) = \kappa(\mathbf{x},t;\mathbf{n},\nu) I(\mathbf{x},t;\mathbf{n},\nu)$$

Moments of the RT equation

- Radiative transfer equation

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{n}\cdot\nabla\right) I(\mathbf{x}, t; \cancel{\mathbf{n}, \nu}) = \eta(\mathbf{x}, t; \cancel{\mathbf{n}, \nu}) - \chi(\mathbf{x}, t; \cancel{\mathbf{n}, \nu}) I(\mathbf{x}, t; \cancel{\mathbf{n}, \nu})$$

TOO HEAVY for multidimensional dynamical calculations

- Zeroth-moment

$$\int d\Omega \times \qquad \frac{\partial E_\nu}{\partial t} + \nabla \cdot \mathbf{F}_\nu = \kappa_\nu (4\pi B_\nu - cE_\nu)$$

- First-moment

$$\int \mathbf{n} d\Omega \times \qquad \frac{1}{c}\frac{\partial \mathbf{F}_\nu}{\partial t} + c\nabla \cdot \mathbb{P}_\nu = -\kappa_\nu \mathbf{F}_\nu$$

Moments models

- System of two equations, three variables => need a closure relation

$$\left\{\begin{array}{l} \dfrac{\partial E_\nu}{\partial t} + \nabla\cdot\mathbf{F}_\nu = \kappa_\nu(4\pi B_\nu - cE_\nu) \\ \dfrac{1}{c}\dfrac{\partial \mathbf{F}_\nu}{\partial t} + c\nabla\cdot\mathbb{P}_\nu = -\kappa_\nu\mathbf{F}_\nu \end{array}\right.$$

- Flux-Limited Diffusion (FLD)

  - Optically thick medium <=> diffusion approximation. Radiation field is isotropic $\mathbb{P}_\nu = \frac{1}{3}\mathbb{I}E_\nu$ and radiative flux is stationary.

$$\mathbf{F}_\nu = -\frac{c}{3\kappa_\nu}\nabla E_\nu$$

$$\mathbf{F}_\nu = -\frac{c\lambda}{\kappa_\nu}\nabla E_\nu$$

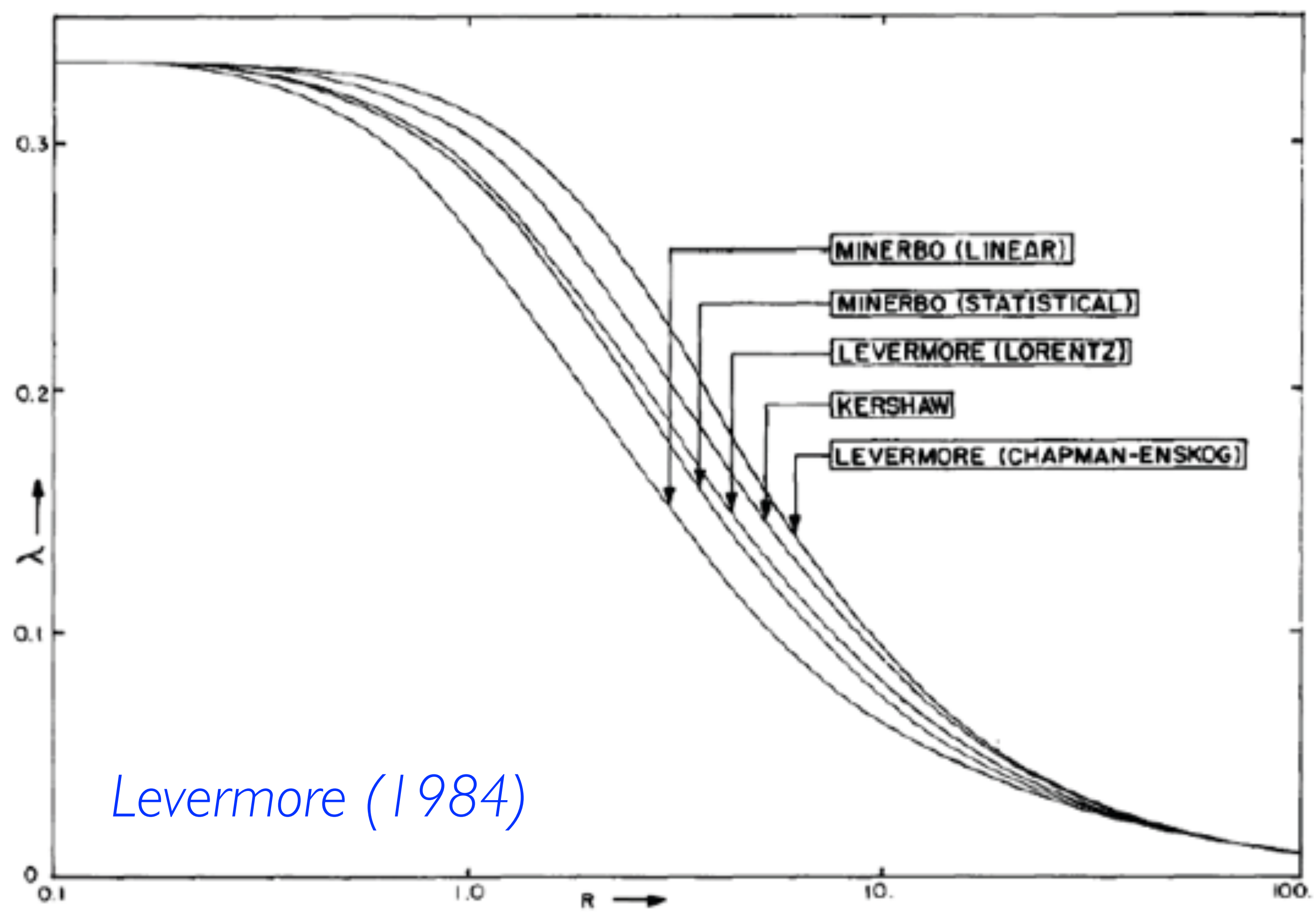
$$\frac{\partial E_\nu}{\partial t} - \nabla\cdot\frac{c\lambda}{\kappa_\nu}\nabla E_\nu = \kappa_\nu(4\pi B_\nu - cE_\nu)$$

Flux Limited Diffusion

- Flux limiter guarantees the two limits for radiation transport
  - ✓ λ is a function of $R = |\nabla E_\nu| / (\kappa_\nu E_\nu)$
  - ✓ λ→1/R in the free streaming limit such that $||\mathbf{F}_\nu|| = cE_\nu$
  - ✓ λ→1/3 in the diffusion limit
- Various flux limiter:

  Minerbo (1978)

  Levermore & Pomraning (1981)



- Eddington tensor $\mathbb{P}_\nu = \mathbb{D} E_\nu$

$$\mathbb{D} = \frac{1-\chi}{2}\mathbb{I} + \frac{3\chi - 1}{2}\mathbf{n} \otimes \mathbf{n} \qquad \chi = \lambda + \lambda^2 R^2$$

Levermore (1984)

Grey Flux Limited Diffusion

- Integration of all radiative quantities over frequency $E_{\rm r} = \int E_\nu d\nu$

$$\frac{\partial E_{\rm r}}{\partial t} - \nabla \cdot \frac{c\lambda}{\kappa_{\rm R}} \nabla E_{\rm r} = \kappa_{\rm P}(a_{\rm r} T^4 - cE_{\rm r})$$

- Planck mean opacity $$\kappa_{\rm P} = \frac{\int \kappa_\nu B_\nu(T) d\nu}{\int B_\nu d\nu}$$

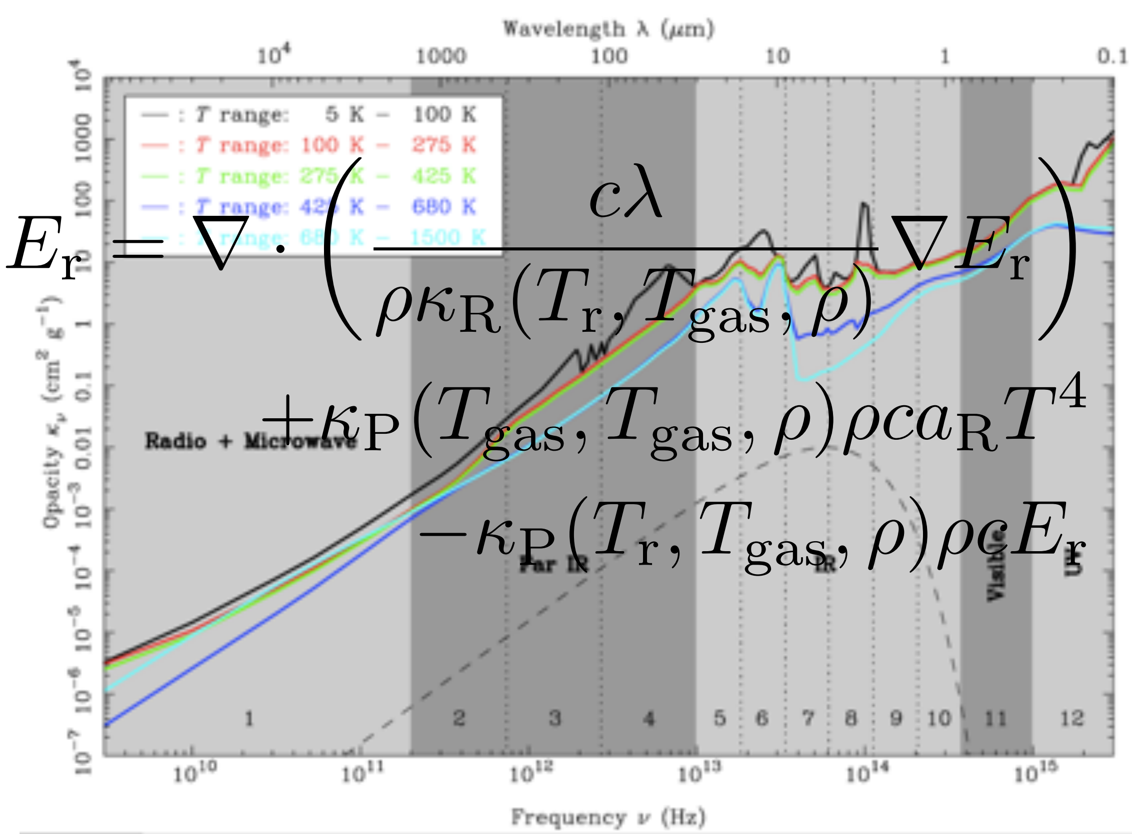
- Rosseland mean opacity $$\frac{1}{\kappa_{\rm R}} = \frac{\int \frac{1}{\kappa_\nu} \frac{\partial B_\nu(T)}{\partial T} d\nu}{\frac{\partial B_\nu(T)}{\partial T} d\nu}$$

Short note on opacity weighting

$$\frac{\partial E_{\rm r}}{\partial t} - \nabla \cdot \frac{c\lambda}{\kappa_{\rm R}} \nabla E_{\rm r} = \kappa_{\rm P}(a_{\rm r}T^4 - cE_{\rm r})$$

✓ usual Rosseland and Planck mean opacities $\kappa_{\rm P} = \frac{\int_{\nu_{\min}}^{\nu_{\max}} \kappa_\nu B_\nu(T,\rho)}{\int_{\nu_{\min}}^{\nu_{\max}} B_\nu(T,\rho)}$

✓ Opacity depends on temperature and density $\kappa_{\rm P} = \frac{\int_{\nu_{\min}}^{\nu_{\max}} \kappa_\nu(T_{\rm gas},\rho) B_\nu(T_{\rm r})}{\int_{\nu_{\min}}^{\nu_{\max}} B_\nu(T_{\rm r})}$



$$\partial_t E_{\rm r} = \nabla \cdot \left(\frac{c\lambda}{\rho \kappa_{\rm R}(T_{\rm r}, T_{\rm gas}, \rho)} \nabla E_{\rm r} \right) + \kappa_{\rm P}(T_{\rm gas}, T_{\rm gas}, \rho)\rho c a_{\rm R} T^4 - \kappa_{\rm P}(T_{\rm r}, T_{\rm gas}, \rho)\rho c E_{\rm r}$$

Vaytet et al. (2012)

But not enough if radiative feedback from stellar sources

$$\partial_t E_{\rm r} = L_\star$$

Lost of spectral information ($T_{{\rm eff},\bigstar} >> T_{\rm r}$)

=> underestimate opacity

(See for instance Kuiper et al. work)

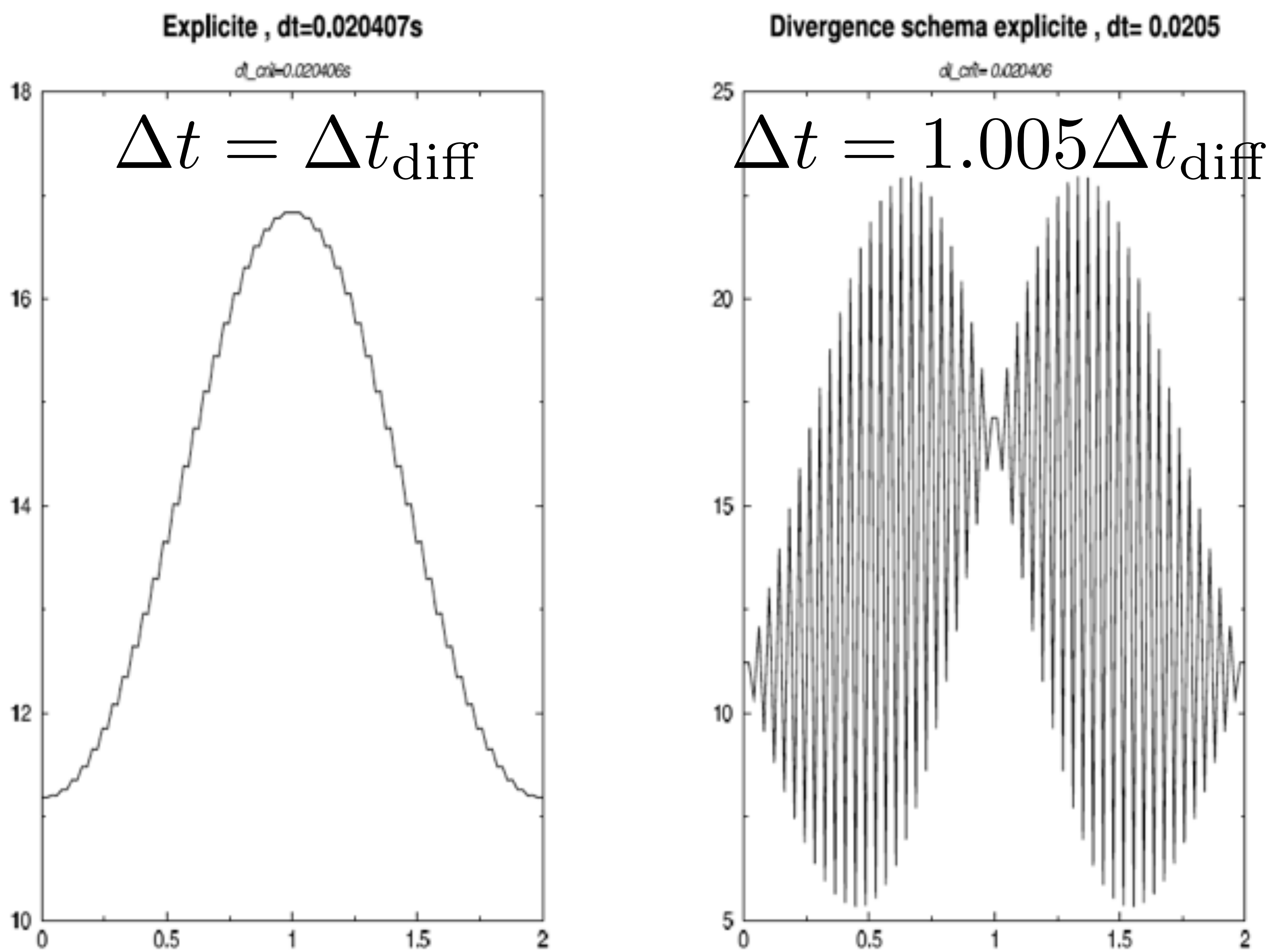
Implicit vs. explicit

- Heat equation $$\frac{\partial E_{\rm r}}{\partial t} = \nabla . K \nabla E_{\rm r}$$
- Explicit discretization $$\frac{E^{n+1}_{{\rm r},i} - E^{n}_{{\rm r},i}}{\Delta t} = K \frac{E^{n}_{{\rm r},i+1} - 2E^{n}_{{\rm r},i} + E^{n}_{{\rm r},i-1}}{\Delta x^2}$$
- Truncation error $$TE = \frac{\Delta t}{2}\frac{\partial^2 E_{\rm r}}{\partial t^2} - K \frac{\Delta x^2}{12}\frac{\partial^4 E_{\rm r}}{\partial t^4} + 0(\Delta t^3, \Delta x^5)$$
- Stability criterion $$\Delta t_{\rm diff} < \frac{\Delta x^2}{2K}$$
- Convergence = Consistency + Stability

Implicit vs. explicit

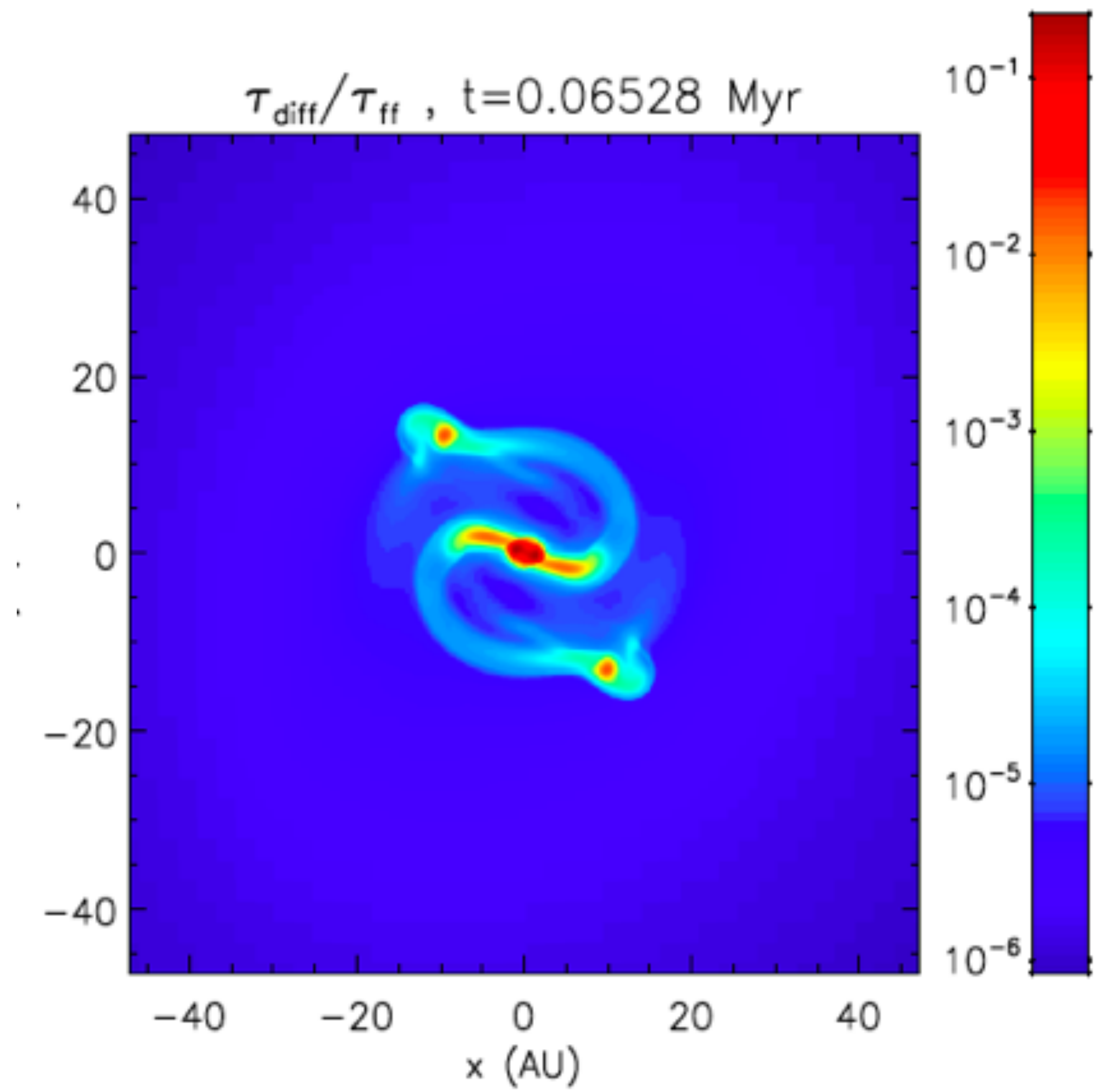
- Explicit scheme

$$\frac{E_{\mathrm{r},i}^{n+1} - E_{\mathrm{r},i}^{n}}{\Delta t} = K \frac{E_{\mathrm{r},i+1}^{n} - 2E_{\mathrm{r},i}^{n} + E_{\mathrm{r},i-1}^{n}}{\Delta x^2}$$



Implicit vs. explicit

Ratio between diffusion time and CFL time in a collapsing core



Stability criterion for parabolic equation

$$\Delta t_{\rm diff} = \frac{\Delta x^2}{2K}$$

Stability criterion for hyperbolic equation

$$\Delta t_{\rm hyd} = C_{\rm CFL}\frac{\Delta x}{v}$$

Implicit vs. explicit

- Heat equation $$\frac{\partial E_{\rm r}}{\partial t} = \nabla . K \nabla E_{\rm r}$$

- Implict discretization $$\frac{E_{{\rm r},i}^{n+1} - E_{{\rm r},i}^{n}}{\Delta t} = K \frac{E_{{\rm r},i+1}^{n+1} - 2E_{{\rm r},i}^{n+1} + E_{{\rm r},i-1}^{n+1}}{\Delta x^2}$$

- Truncation error $$TE = \frac{\Delta t}{2}\frac{\partial^2 E_{\rm r}}{\partial t^2} - K\frac{\Delta x^2}{12}\frac{\partial^4 E_{\rm r}}{\partial t^4} + 0(\Delta t^3, \Delta x^5)$$

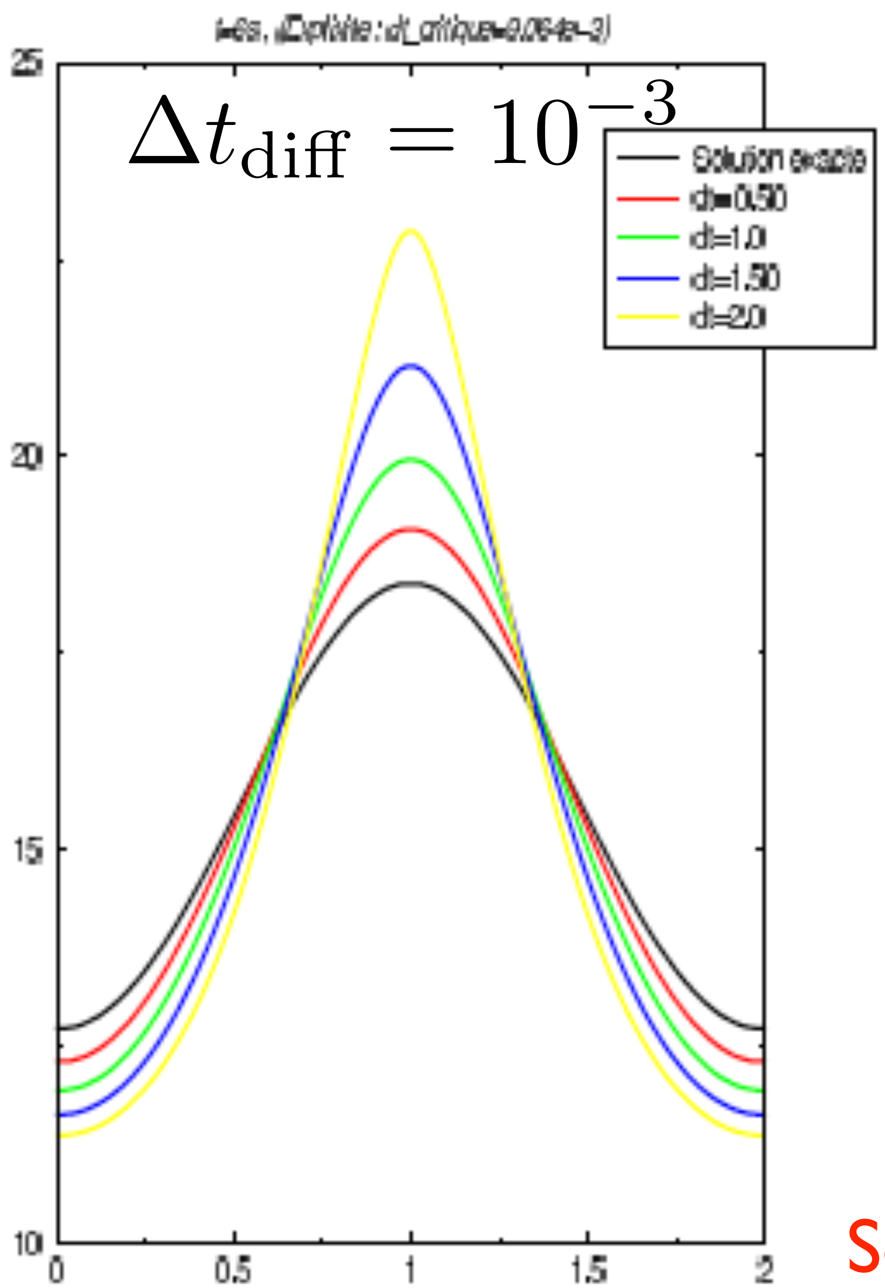
- Unconditionnaly stable

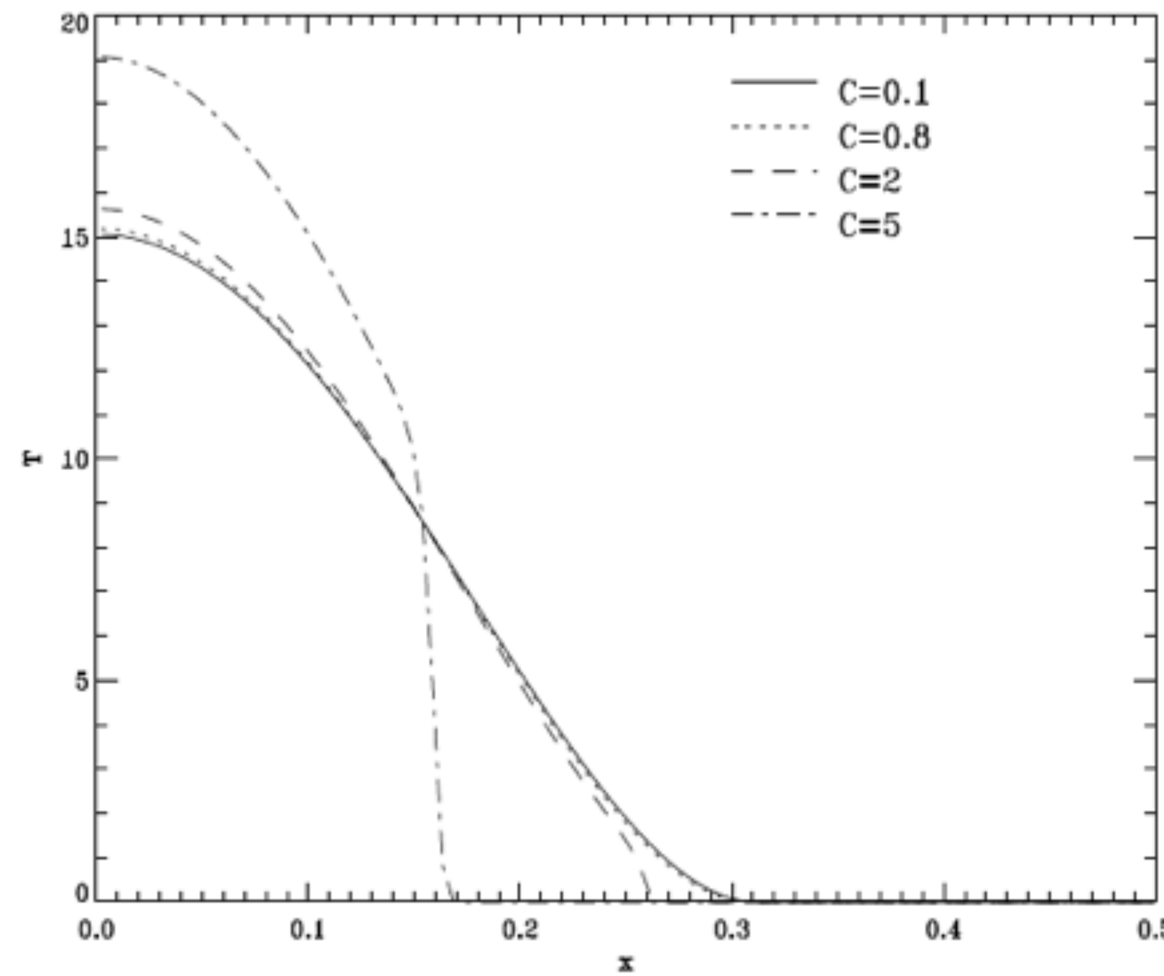
➡ Choice of the implicit scheme for diffusion terms for the rest of the lecture

Implicit vs. explicit

- Implicit scheme

$$\frac{E_{r,i}^{n+1} - E_{r,i}^{n}}{\Delta t} = K \frac{E_{r,i+1}^{n+1} - 2E_{r,i}^{n+1} + E_{r,i-1}^{n+1}}{\Delta x^2}$$





Non-linear diffusion coefficient

Scheme is convergent but large truncation errors

Which frame?

- Radiative quantities are estimated in the laboratory frame
- but… matter/radiation interactions are estimated in the comoving frame
- Fluid equations are estimated in the comoving frame.

➡ Let's go for comoving

- Lorentz transformation from the laboratory to the fluid frame at 0(1) in v/c

see e.g. Mihalas & Klein (1982), Krumholz et al. (2007) for a mixed frame formulation

$$\begin{cases} \dfrac{\partial E_\nu}{\partial t} + \nabla \cdot [\mathbf{u} E_\nu] + \nabla \cdot \mathbf{F}_\nu + \mathbb{P}_\nu : \nabla \mathbf{u} = \kappa_\nu (4\pi B_\nu - c E_\nu) \\ \dfrac{\partial \mathbf{F}_\nu}{\partial t} + \nabla \cdot [\mathbf{u} \mathbf{F}_\nu] + c^2 \nabla \cdot \mathbb{P}_\nu + (\mathbf{F}_\nu \cdot \nabla) \mathbf{u} = -\kappa_\nu \mathbf{F}_\nu \end{cases}$$

FLD - RHD equations

- System of 4 equations, mix hyperbolic and parabolic

$$\left\{\begin{array}{lcl}
\partial_t \rho + \nabla \cdot [\rho \mathbf{u}] & = & 0 \\
\partial_t \rho \mathbf{u} + \nabla \cdot [\rho \mathbf{u} \otimes \mathbf{u} + P\mathbb{I}] & = & -\lambda \nabla E_{\mathrm{r}} \\
\partial_t E_{\mathrm{T}} + \nabla \cdot [\mathbf{u}\,(E_{\mathrm{T}} + P)] & = & -\mathbb{P}_{\mathrm{r}} \nabla : \mathbf{u} - \lambda \mathbf{u} \nabla E_{\mathrm{r}} \\
 & & +\nabla \cdot \left(\frac{c\lambda}{\rho \kappa_{\mathrm{R}}} \nabla E_{\mathrm{r}} \right) \\
\partial_t E_{\mathrm{r}} + \nabla \cdot [\mathbf{u} E_{\mathrm{r}}] & = & -\mathbb{P}_{\mathrm{r}} \nabla : \mathbf{u} + \nabla \cdot \left(\frac{c\lambda}{\rho \kappa_{\mathrm{R}}} \nabla E_{\mathrm{r}} \right) \\
 & & +\kappa_{\mathrm{P}} \rho c (a_{\mathrm{R}} T^4 - E_{\mathrm{r}}),
\end{array}\right.$$

$$E_{\mathrm{T}} = \frac{P}{\gamma - 1} + \rho \frac{\mathbf{u}^2}{2} + E_{\mathrm{r}}$$

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2. **RHD with Grey Flux Limited Diffusion**
    - integration in RAMSES
    - adaptive time-steps
    - tests
3. Multigroup FLD
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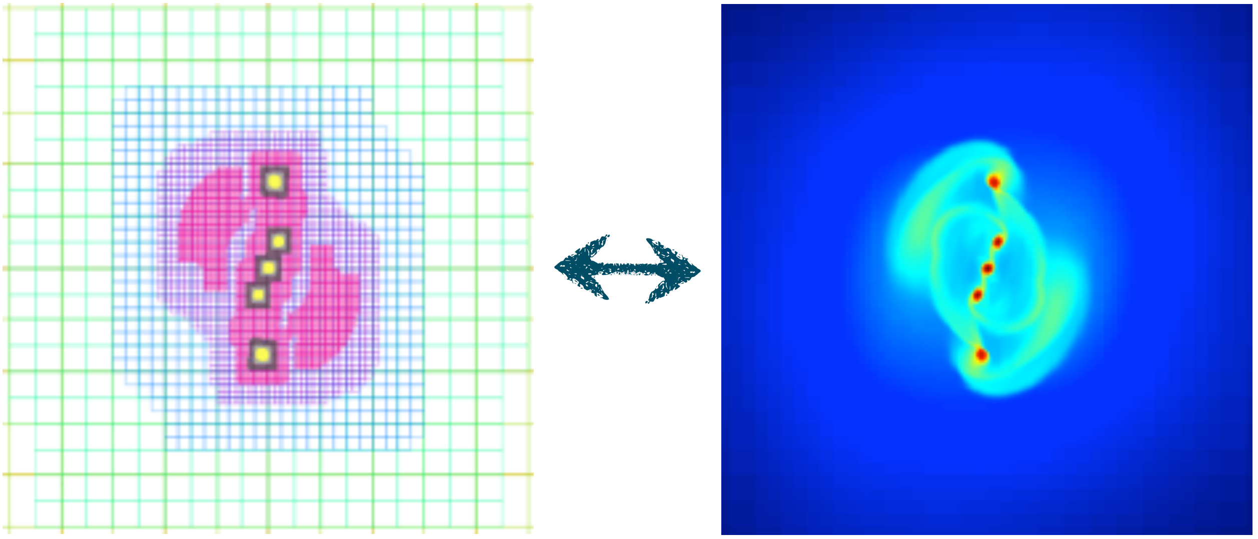
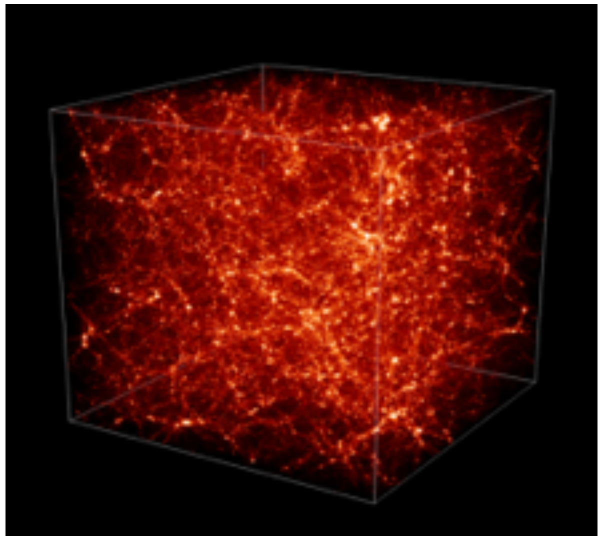
4. Extension to cosmic rays hydrodynamics

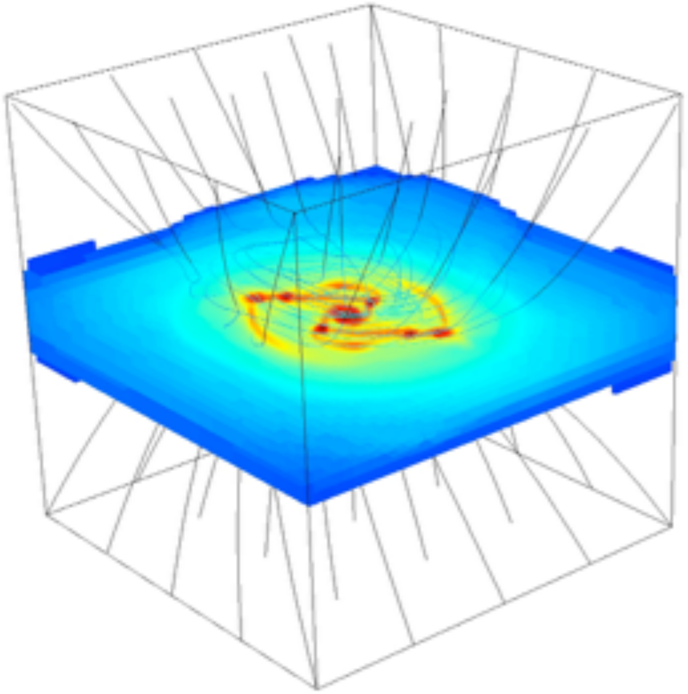
RAMSES code

✓ **RAMSES code** *(Teyssier 2002)*

- Adaptive Mesh Refinement - cell by cell
- 2nd order Godunov finite volume
- MUSCL-Hancock predictor/corrector
- adaptive time-steps
- MPI parallel
- ideal and non-ideal MHD (*Fromang et al., Teyssier et al. 2006, Masson et al. 2012*)
- sink particles using clump finder (*Bleuler & Teyssier 2014*)







RHD with Flux Limited Diffusion in RAMSES

- ✓ **RHD** solver in the *comoving* frame using the grey Flux Limited Diffusion approximation (*Commerçon et al. 2011a, 2014*)

$$\begin{cases} \partial_t \rho + \nabla[\rho \mathbf{u}] = 0 \\ \partial_t \rho \mathbf{u} + \nabla[\rho \mathbf{u} \otimes \mathbf{u} + (P + 1/3 E_r)\mathbb{I}] = -(\lambda - 1/3)\nabla E_r \\ \partial_t E_T + \nabla[\mathbf{u}(E_T + P + 1/3 E_r)] = -(\lambda - 1/3)\nabla(\mathbf{u}E_r) + \nabla \cdot \left(\frac{c\lambda}{\rho\kappa_R}\nabla E_r\right) \\ \partial_t E_r + \nabla[\mathbf{u}E_r] = -\mathbb{P}_r : \nabla \mathbf{u} + \kappa_P \rho c(a_R T^4 - E_r) + \nabla \cdot \left(\frac{c\lambda}{\rho\kappa_R}\nabla E_r\right) \end{cases}$$

Riemann solver - explicit | Corrective terms - explicit | Coupling + Diffusion - implicit

$$\partial t\mathbb{U} + \nabla\mathbb{F}(\mathbb{U}) = \mathbb{S}(\mathbb{U})$$

$$\mathbb{P} = \begin{bmatrix} \rho \\ \mathbf{u} \\ P \\ E_r \end{bmatrix} \quad \mathbb{U} = \begin{bmatrix} \rho \\ \rho\mathbf{u} \\ E_T \\ E_r \end{bmatrix}$$

Primitive   Conservative

$$\Delta t \leq C_{\mathrm{CFL}} \frac{\Delta x}{u + \sqrt{\frac{\gamma P}{\rho} + \frac{4E_r}{9\rho}}}$$

Godunov part

$$\partial t\mathbb{U} + \nabla\mathbb{F}(\mathbb{U}) = 0 \qquad \qquad \partial t\mathbb{P} + \mathbb{B}(\mathbb{P})\nabla\mathbb{P} = 0$$

- Jacobian matrix

$$\mathbb{B}(\mathbb{V}) = \begin{pmatrix} u & \rho & 0 & 0 \\ 0 & u & \frac{1}{\rho} & \frac{1}{3\rho} \\ 0 & \gamma P & u & 0 \\ 0 & \frac{4E_{\mathrm{r}}}{3} & 0 & u \end{pmatrix}$$

- 3 eigenvalues

✓ Largest fan of solution

$$\lambda_i = \begin{cases} u - \sqrt{\frac{\gamma P}{\rho} + \frac{4E_{\mathrm{r}}}{9\rho}} \\ u \\ u + \sqrt{\frac{\gamma P}{\rho} + \frac{4E_{\mathrm{r}}}{9\rho}} \end{cases}$$

Godunov part

$$\partial t\mathbb{U} + \nabla\mathbb{F}(\mathbb{U}) = 0 \quad \longrightarrow \quad \partial t\mathbb{P} + \mathbb{B}(\mathbb{P})\nabla\mathbb{P} = 0$$

- Jacobian matrix

$$\mathbb{B}(\mathbb{V}) = \begin{pmatrix} u & \rho & 0 & 0 \\ 0 & u & \frac{1}{\rho} & \frac{1}{3\rho} \\ 0 & \gamma P & u & 0 \\ 0 & \frac{4E_\mathrm{r}}{3} & 0 & u \end{pmatrix}$$

- update the flux using

approximate Riemann solvers (LF, HLL, HLLD)

$$\mathbb{F}(\mathbb{U}) = \begin{bmatrix} \rho\mathbf{u} \\ \rho\mathbf{u}\otimes\mathbf{u} + (P + 1/3E_\mathrm{r})\mathbb{I} \\ \mathbf{u}\left(E_\mathrm{T} + P + 1/3E_\mathrm{r}\right) \\ \mathbf{u}E_\mathrm{r} \end{bmatrix}$$

Same can be done for any other non-thermal energy (e.g. cosmic rays)

Source terms

- Correct for too large radiative pressure in radiative force (momentum) and radiative pressure work (total energy)

+ update radiative energy

$$\mathbb{S}(\mathbb{U}) = \left(\begin{array}{c} 0 \\ -(\lambda - 1/3)\nabla E_{\rm r} \\ -(\lambda - 1/3)(\mathbf{u} \cdot \nabla E_{\rm r} + E_{\rm r} \nabla : \mathbf{u}) \\ \mathbb{P}_{\rm r} \nabla : \mathbf{u} \end{array} \right).$$

- Terms are estimated using finite differences, accounting for AMR grid effects
- Accounts for anisotropy in the radiative pressure tensor

Implicit update of diffusion/coupling

- ✓ Finite volume framework

$$\frac{\Delta E_{\rm r}}{\Delta t} V = F \times S$$

$$\begin{cases} \dfrac{C_v T^{n+1} - C_v T^n}{\Delta t} = & -\kappa_{\rm P}^n \rho^n c(a_{\rm R}(T^{n+1})^4 - E_{\rm r}^{n+1}) \\ \dfrac{E_{\rm r}^{n+1} - E_{\rm r}^n}{\Delta t} - \nabla \dfrac{c\lambda^n}{\kappa_{\rm R}^n \rho^n} \nabla E_{\rm r}^{n+1} = & +\kappa_{\rm P}^n \rho^n c(a_{\rm R}(T^{n+1})^4 - E_{\rm r}^{n+1}), \end{cases}$$

- ✓ Implicit discretization

$$\begin{aligned} (E_{r,i}^{n+1} - E_{r,i}^n) V_i \quad & - \quad c\Delta t \left(\frac{\lambda}{\kappa_{\rm R}}\right)_{i+1/2} S_{i+1/2} \frac{E_{r,i+1}^{n+1} - E_{r,i}^{n+1}}{\Delta x_{i+1/2}} \\ & + \quad c\Delta t \left(\frac{\lambda}{\kappa_{\rm R}}\right)_{i-1/2} S_{i-1/2} \frac{E_{r,i}^{n+1} - E_{r,i-1}^{n+1}}{\Delta x_{i-1/2}} \\ & = \quad c\Delta t \kappa_{{\rm P},i}^n \left(4a_{\rm R}(T_i^n)^3 T_i^{n+1} - 3a_{\rm R}(T_i^n)^4 - E_{r,i}^{n+1}\right) V_i \end{aligned}$$

- ✓ Linearize the emission term

$$(T^{n+1})^4 = (T^n)^4 \left(1 + \frac{(T^{n+1} - T^n)}{T^n}\right)^4 \approx 4(T^n)^3 T^{n+1} - 3(T^n)^4$$

- ✓ Matrix system to invert which is symmetric and positive definite

Implicit update of diffusion/coupling

- ✓ Implicit solved with an iterative conjugate gradient algorithm

$$-C_{i-1/2}E_{\mathrm{r},i-1}^{n+1} + (1 + C_{i-1/2} + C_{i+1/2})E_{\mathrm{r},i}^{n+1} - C_{i+1/2}E_{\mathrm{r},i+1}^{n+1} = f(E_{\mathrm{r},i}^{n}, T_i^{n+1})$$

  - ✓ matrix elements are stored during iterations
  - ✓ diagonal preconditionning
  - ✓ scales in N log(N)

- ✓ Update gas temperature

$$T_i^{n+1} = \frac{3a_{\mathrm{R}}\kappa_{\mathrm{P},i}^{n}c\Delta t\,(T_i^n)^4 + C_v T_i^n + \kappa_{\mathrm{P},i}^{n}c\Delta t E_{r,i}^{n+1}}{C_v + 4a_{\mathrm{R}}\kappa_{\mathrm{P},i}^{n}c\Delta t\,(T_i^n)^3}$$

⚠ Linearisation only works if temperature changes are small

Implicit integration of a diffusion equation

- **Simple heat equation:** $$\frac{\partial E_{\mathrm{r}}}{\partial t} = \nabla . K \nabla E_{\mathrm{r}}$$

✓ Implicit scheme is unconditionally stable

➡ How to speed-up implicit schemes on AMR grids?

✓ use a unique time step for all the levels and couple all the levels (*Commerçon et al. 2011*)

✓ each level evolves "independently" form the others: needs to specify boundary conditions at level interfaces (e.g., nested grids, *Tomida et al.)*

✓ use adaptive time stepping (*ORION, CASTRO*)

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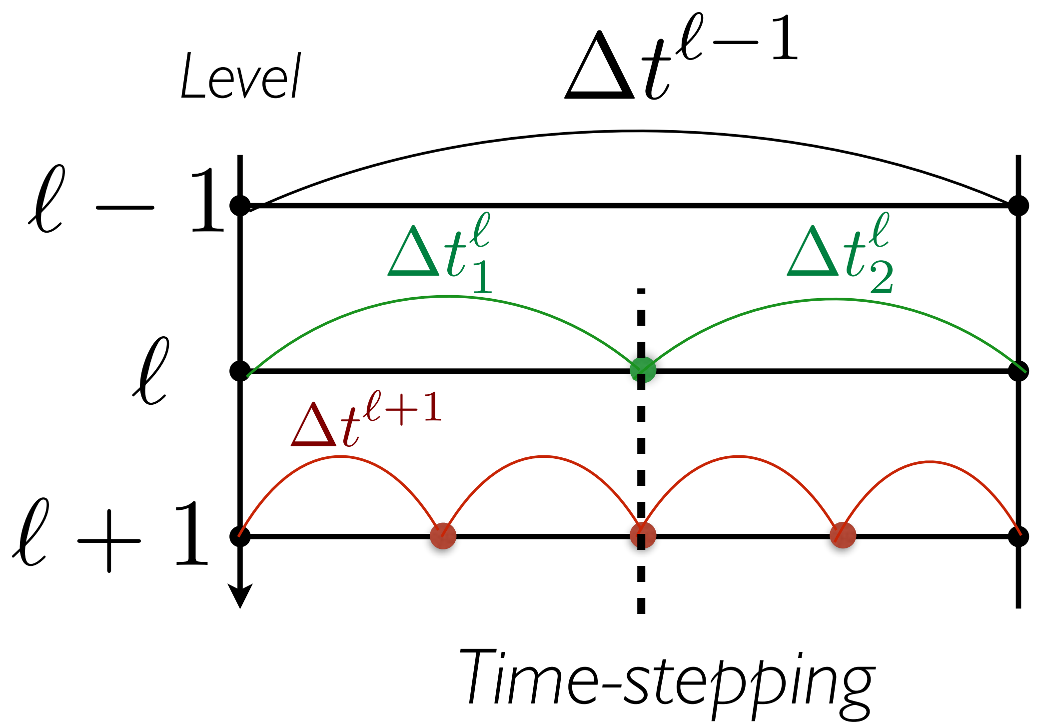
Adaptive time-stepping on AMR grid



Straightforward for explicit scheme at coarse-to-fine interface:

$$F_{i+1/2}^{n+\Delta t^{\ell-1}} = \frac{1}{\Delta t_1^{\ell} + \Delta t_2^{\ell}} \left(\Delta t_1^{\ell} F_{i+1/2}^{n+\Delta t_1^{\ell}} + \Delta t_2^{\ell} F_{i+1/2}^{n+\Delta t_1^{\ell}+\Delta t_2^{\ell}} \right)$$

+ energy is conserved
+ highly efficient for hydrodynamics

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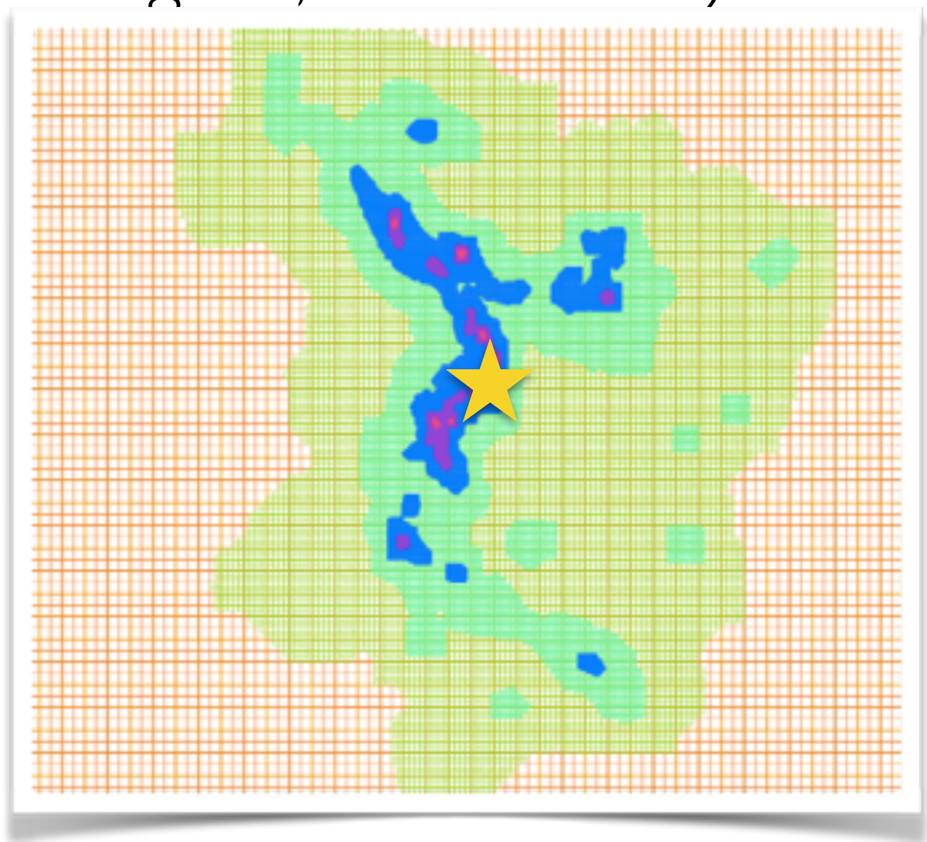
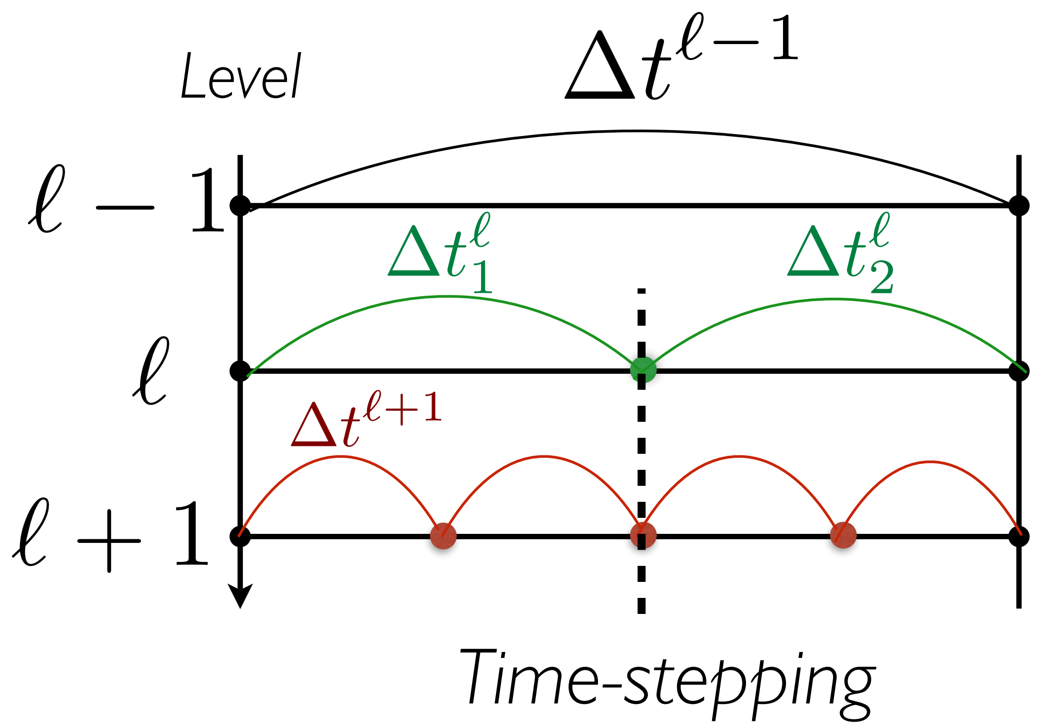
- ✓ use a unique time step for all the levels and couple all the levels (*Commerçon et al. 2011*)
- ✓ each level evolves "independently" form the others: needs to specify boundary conditions at level interfaces (e.g., nested grids, *Tomida et al.*)
- ✓ use adaptive time stepping (*ORION, CASTRO*)



✓ Recurrent problems:
- ✓ energy propagation
- ✓ energy conservation

Adaptive time-stepping on AMR grid



Straightforward for explicit scheme
at coarse-to-fine interface:

$$F_{i+1/2}^{n+\Delta t^{\ell-1}} = \frac{1}{\Delta t_1^{\ell} + \Delta t_2^{\ell}} \left(\Delta t_1^{\ell} F_{i+1/2}^{n+\Delta t_1^{\ell}} + \Delta t_2^{\ell} F_{i+1/2}^{n+\Delta t_1^{\ell}+\Delta t_2^{\ell}} \right)$$

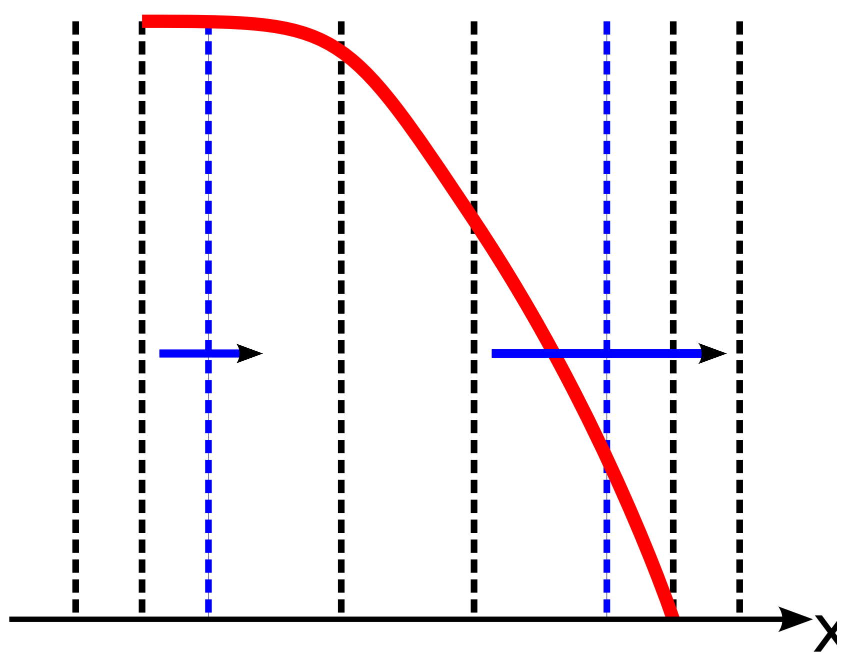
+ energy is conserved
+ highly efficient for hydrodynamics

but….

energy does not propagate more than one cell (CFL condition)

=> What happens for implicit schemes when flux are stored?

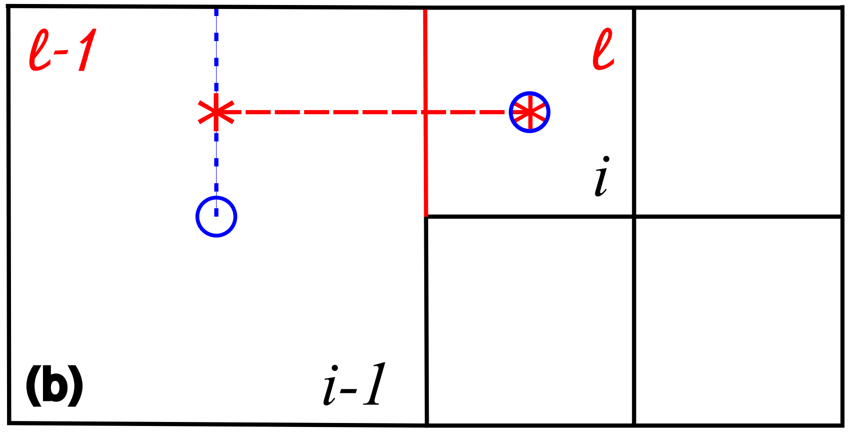
NEGATIVE ENERGY!!!!



Grid configuration and boundary conditions

Commerçon et al. (2014)

- **Dirichlet:** imposed boundary value ($E_r = E_{r,b}$)
  ➡ robust , but energy is not conserved
- **Neumann:** imposed flux condition ($\mathbf{F}_r = \mathbf{F}_{r,b}$)
  ➡ energy is conserved (e.g. *Howell & Greenhough 2003*)
- **Robin:** mix between Dirichlet and Neumann, weighted by a parameter α

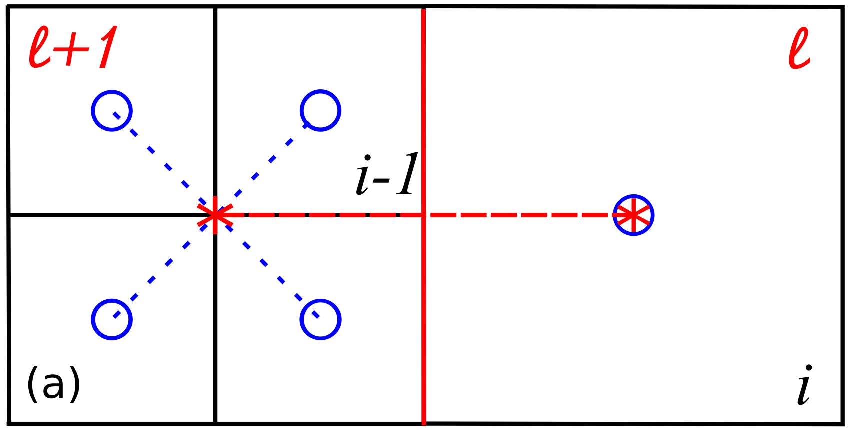


Neighbor is less refined

- **Fine-to-coarse interface:** Dirichlet BC

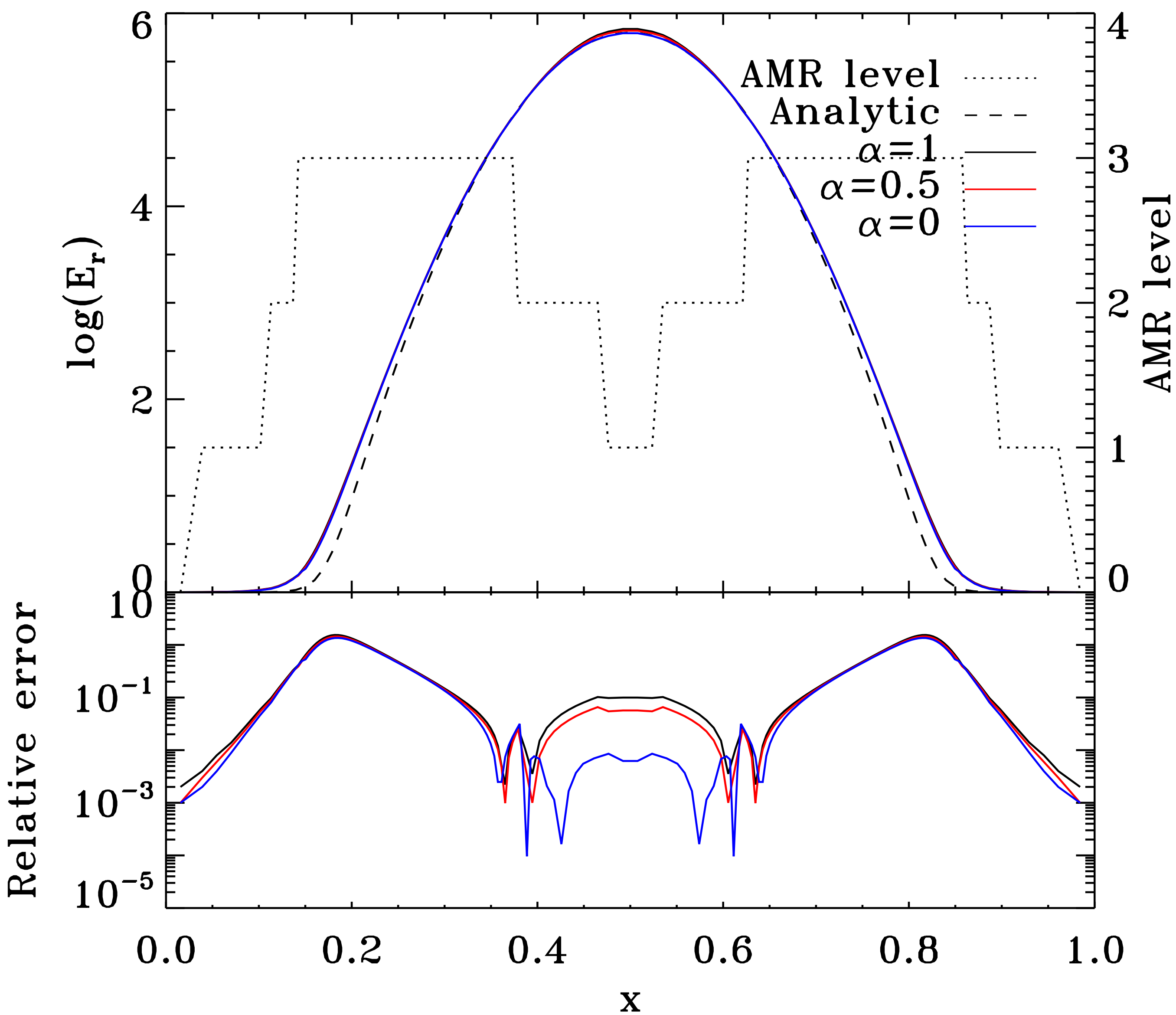
$$\tilde{F}_{i-1/2} = -K_{i-1/2} \frac{E^{n+1}_{r,i} - E^{n}_{r,i-1}}{\frac{3}{2}\Delta x}$$

- **Coarse-to-fine:** 3 possibilities
  …but energy mismatch in Dirichlet and Robin case

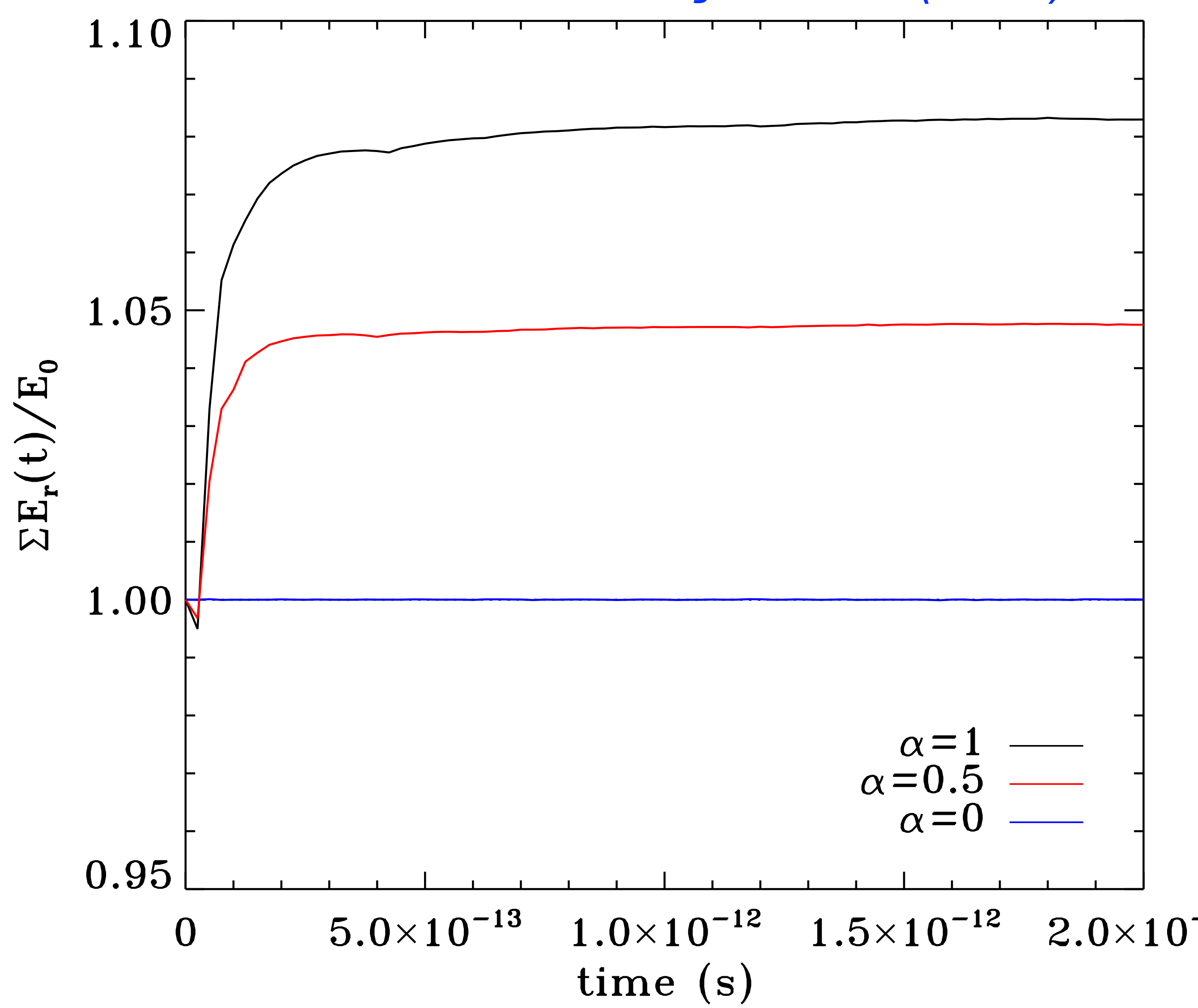


Neighbor is more refined

Test: 1D Dirac diffusion



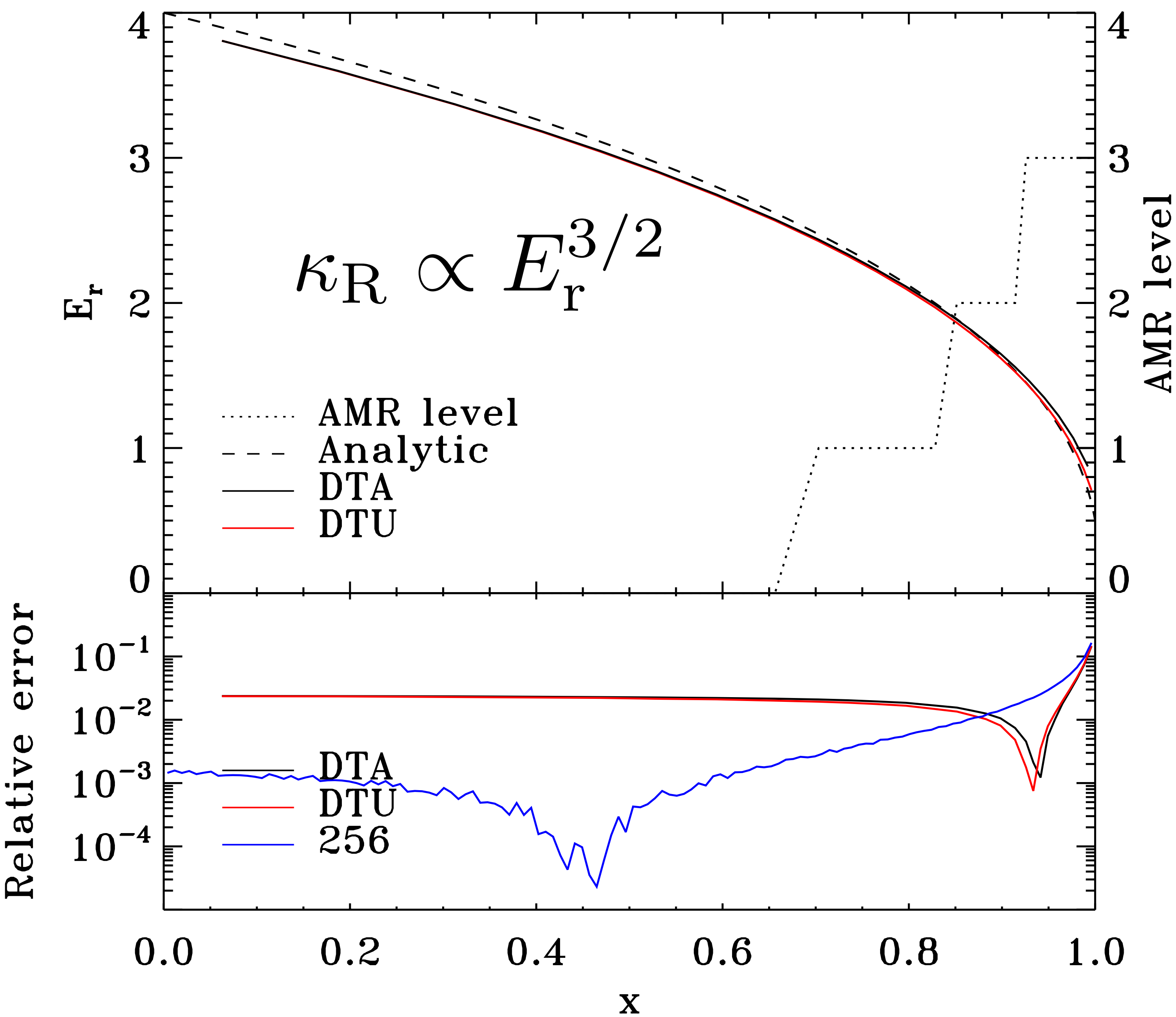
Dirichlet: $\alpha = 1$



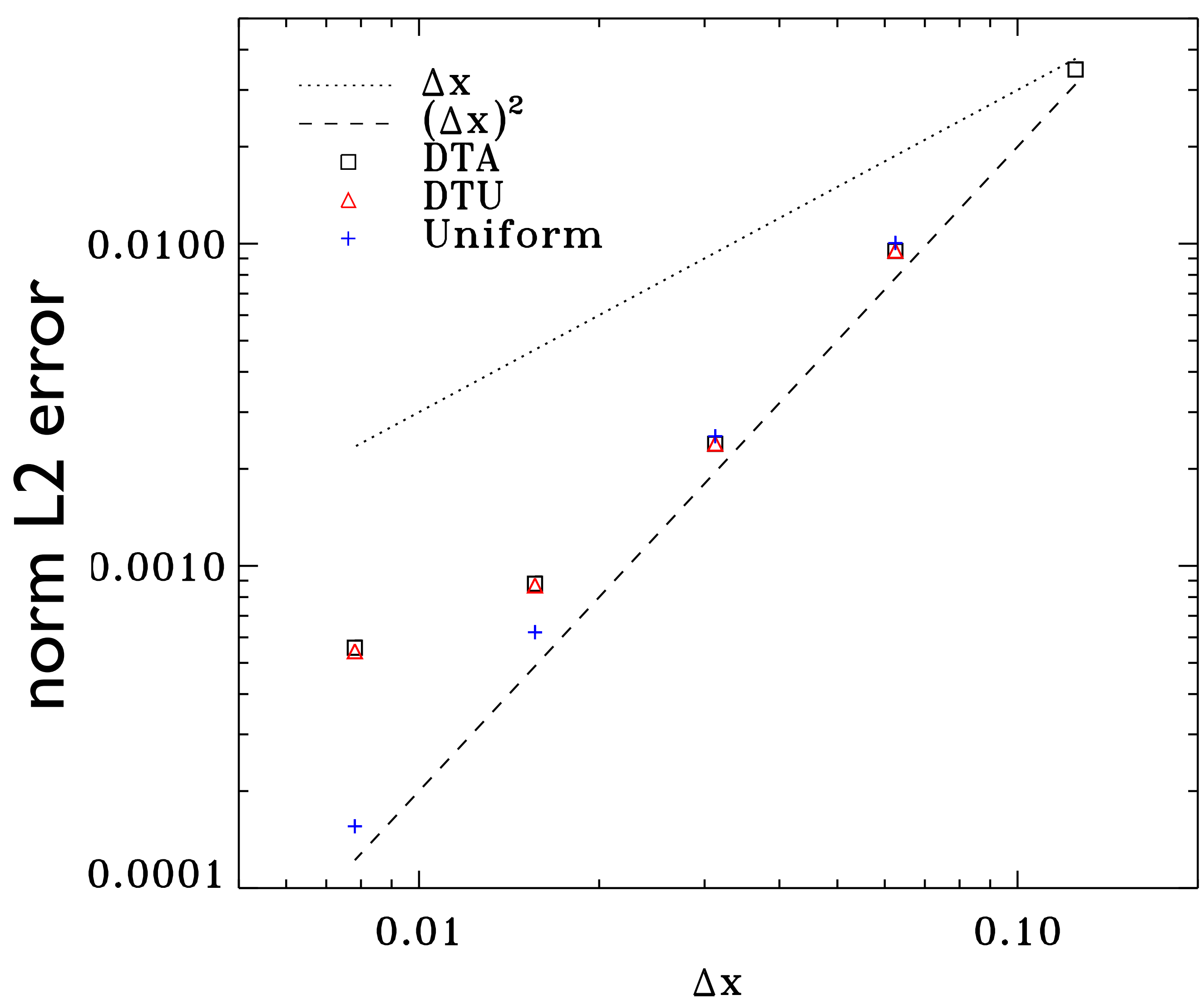
Energy conservation

good results with Dirichlet even if energy is not conserved

Test: stationary non-linear diffusion



Commerçon et al. (2014)



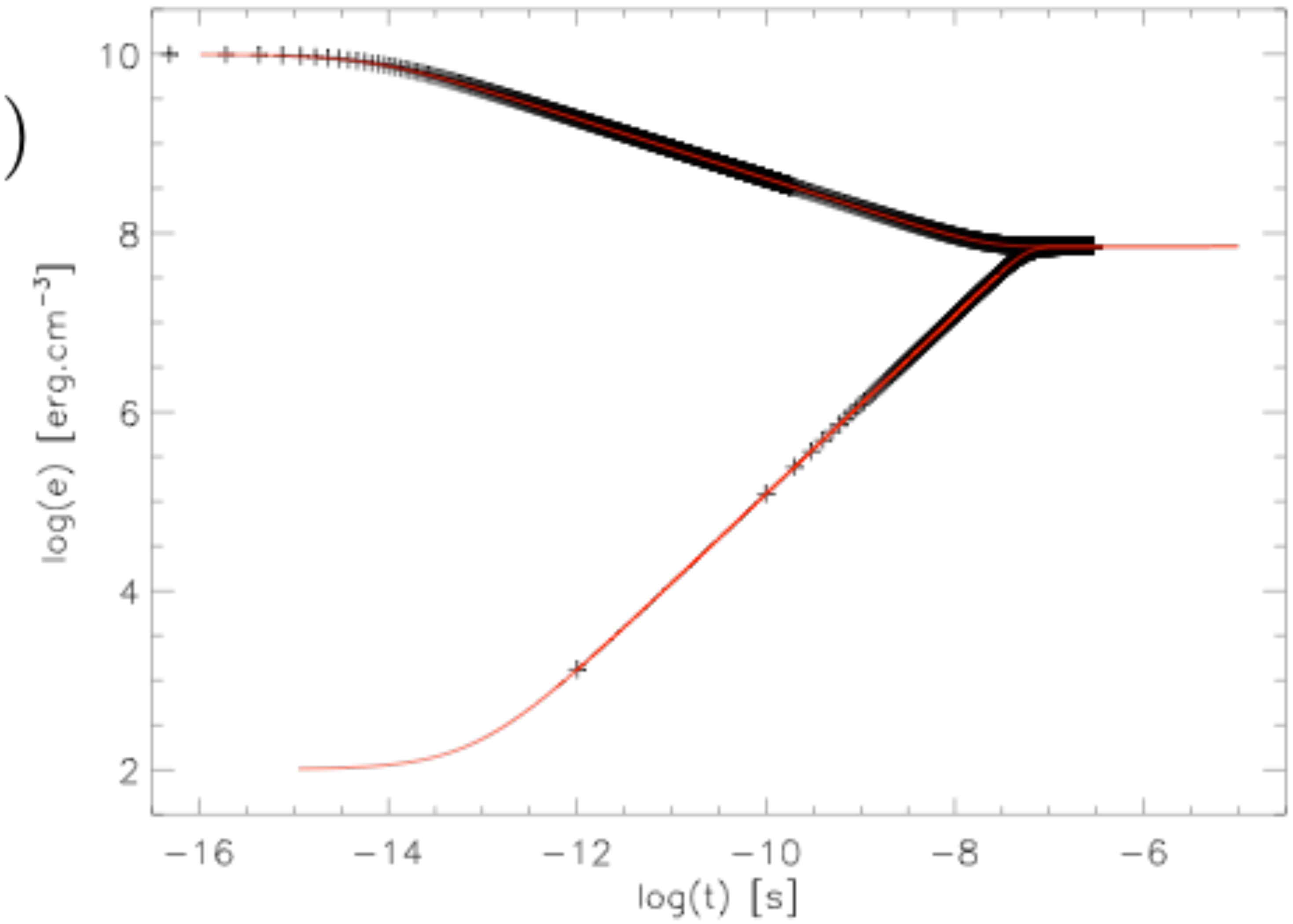
2nd order in space

- similar results than using using a unique time step
- Neumann and Robin BC do not pass this test because of negative energy (initial gradient)…

Test: radiation/matter coupling

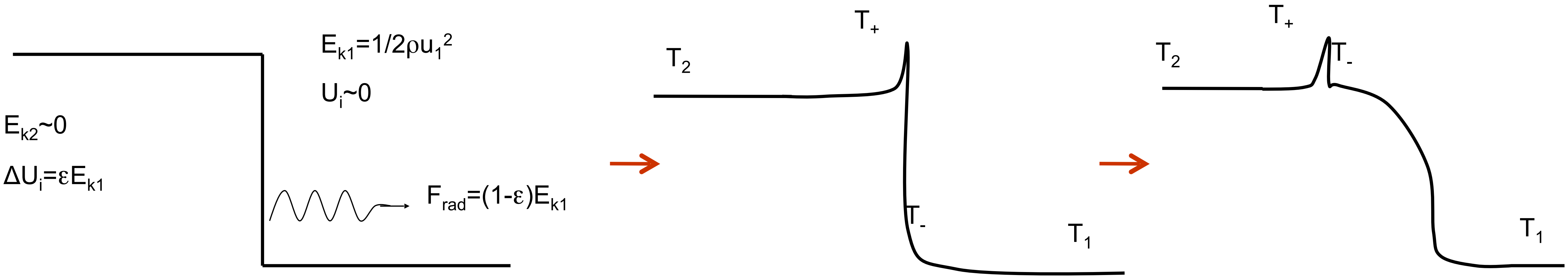
Commerçon et al. (2011)

$$\frac{de}{dt} = c\sigma E_{\rm r} - 4\pi\sigma B(e)$$



- need to relax the time step!

Test: radiative shocks



- Jump conditions (Rankine-Hugoniot)

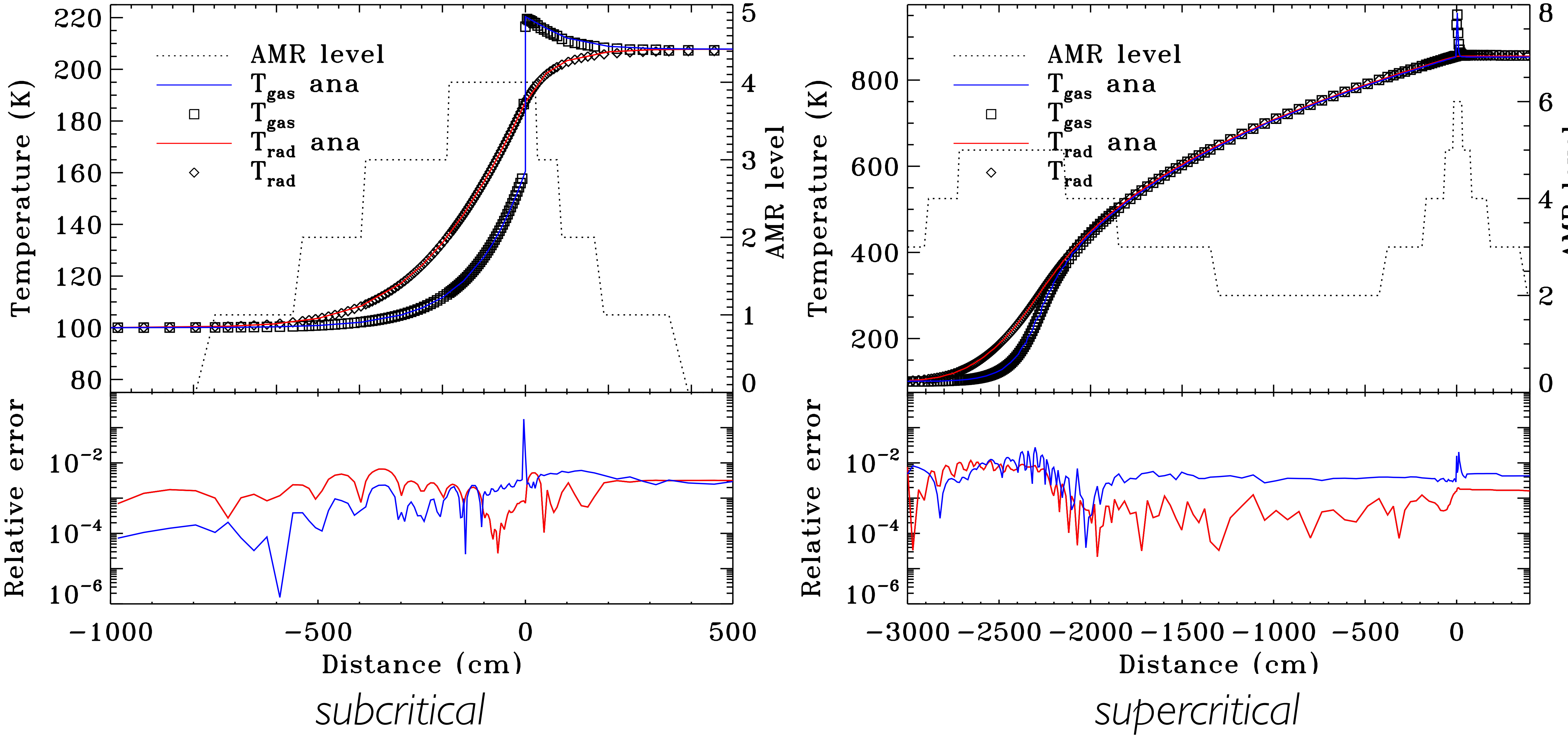
$$\rho_1 u_1 = \rho_2 u_2 \equiv \dot{m},$$
$$\rho_1 u_1^2 + P_1 + P_{r1} = \rho_2 u_2^2 + P_2 + P_{r2},$$
$$\dot{m}\left(h_1 + \rho_1 u_1^2\right) + F_{r1} + u_1\left(E_{r1} + P_{r1}\right) =$$
$$\dot{m}\left(h_2 + \rho_2 u_2^2\right) + F_{r2} + u_2\left(E_{r2} + P_{r2}\right)$$

- Shock becomes supercritical if $T_- > T_{cr}$

$$T_{\rm cr} = \left(\frac{u_1 \rho_1 k_{\rm B}}{(\gamma - 1)\mu m_{\rm H}\sigma}\right)^{1/3}$$

Commerçon et al. (2011b)

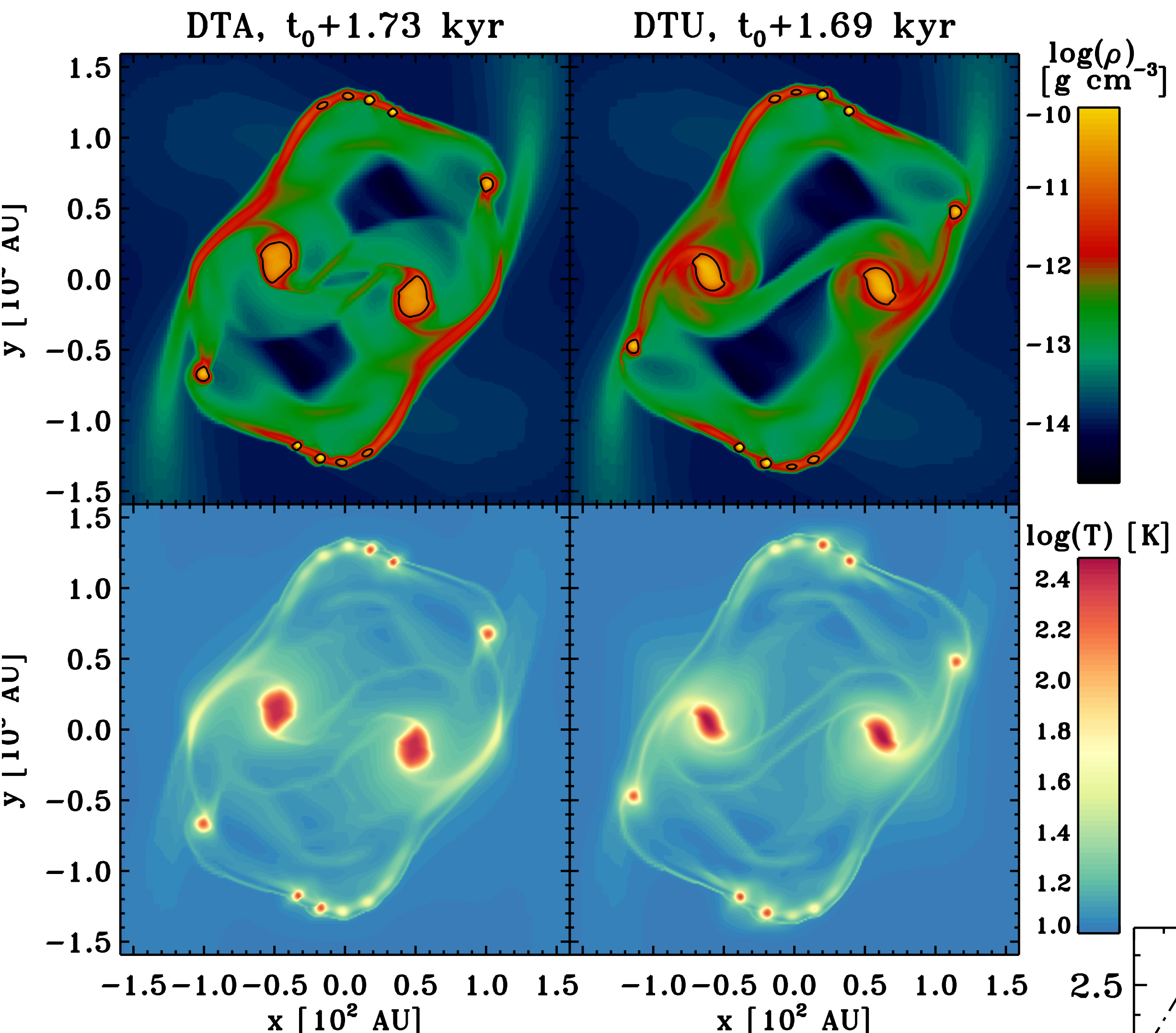
Test: radiative shocks

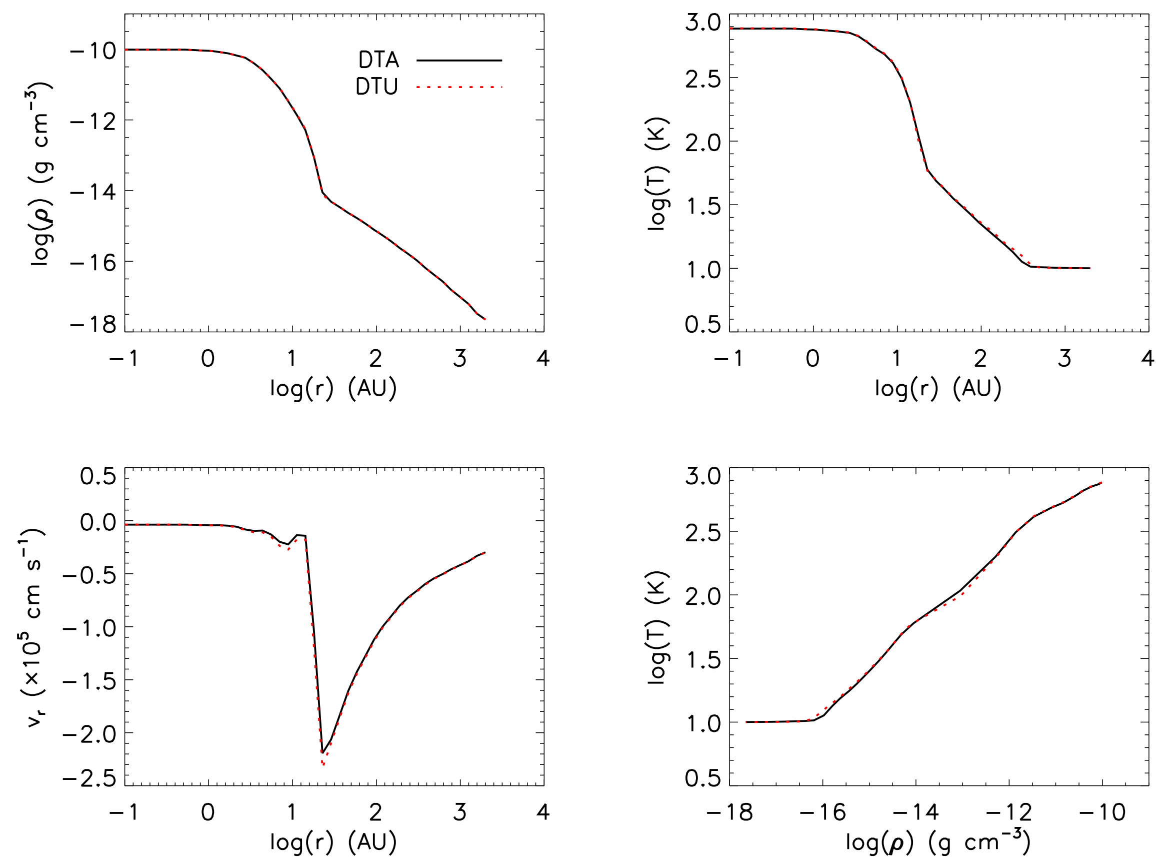


Commerçon et al. (2014)

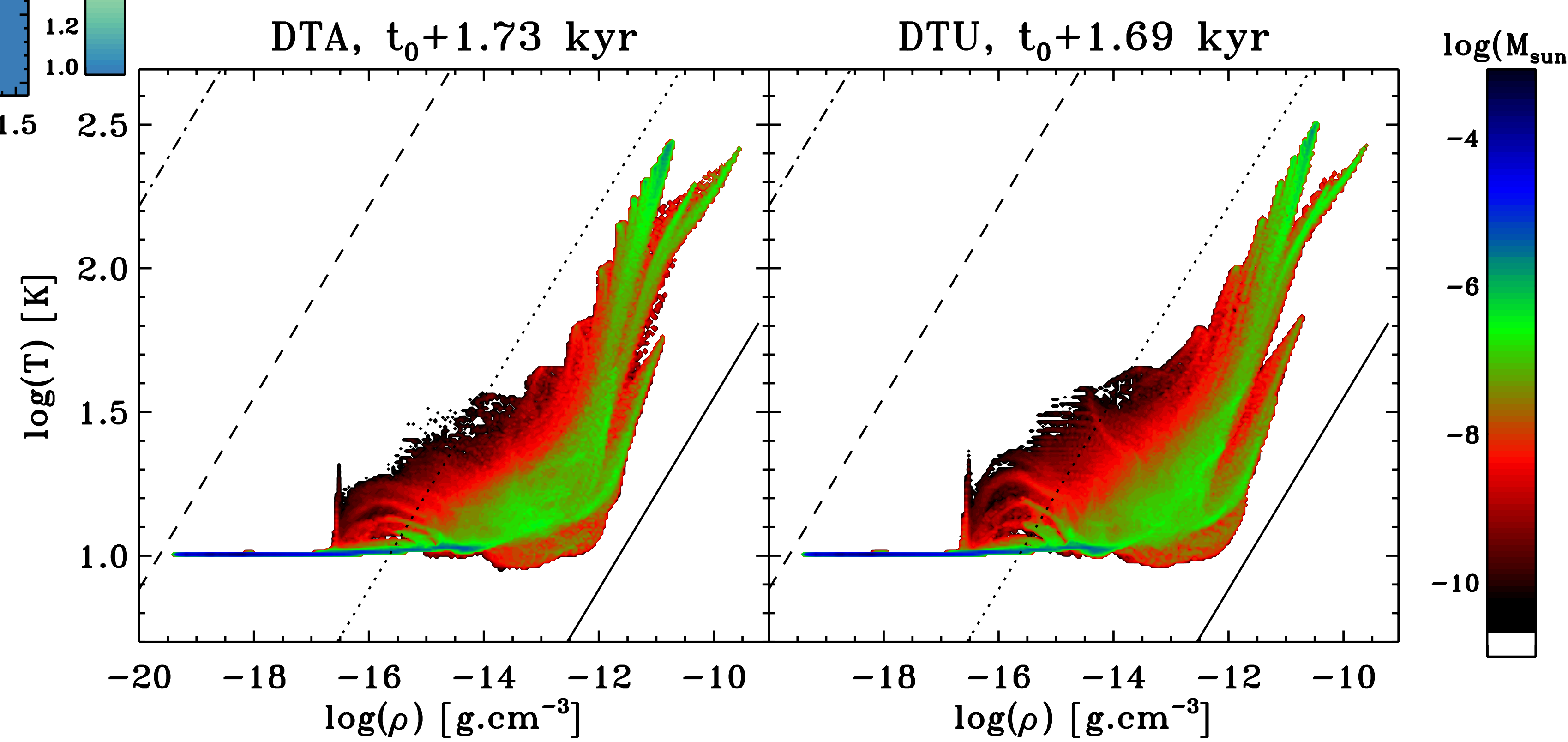
Gain in CPU time compared to synchronised timestep: 50-100

Test 3D: 1 solar mass dense core collapse



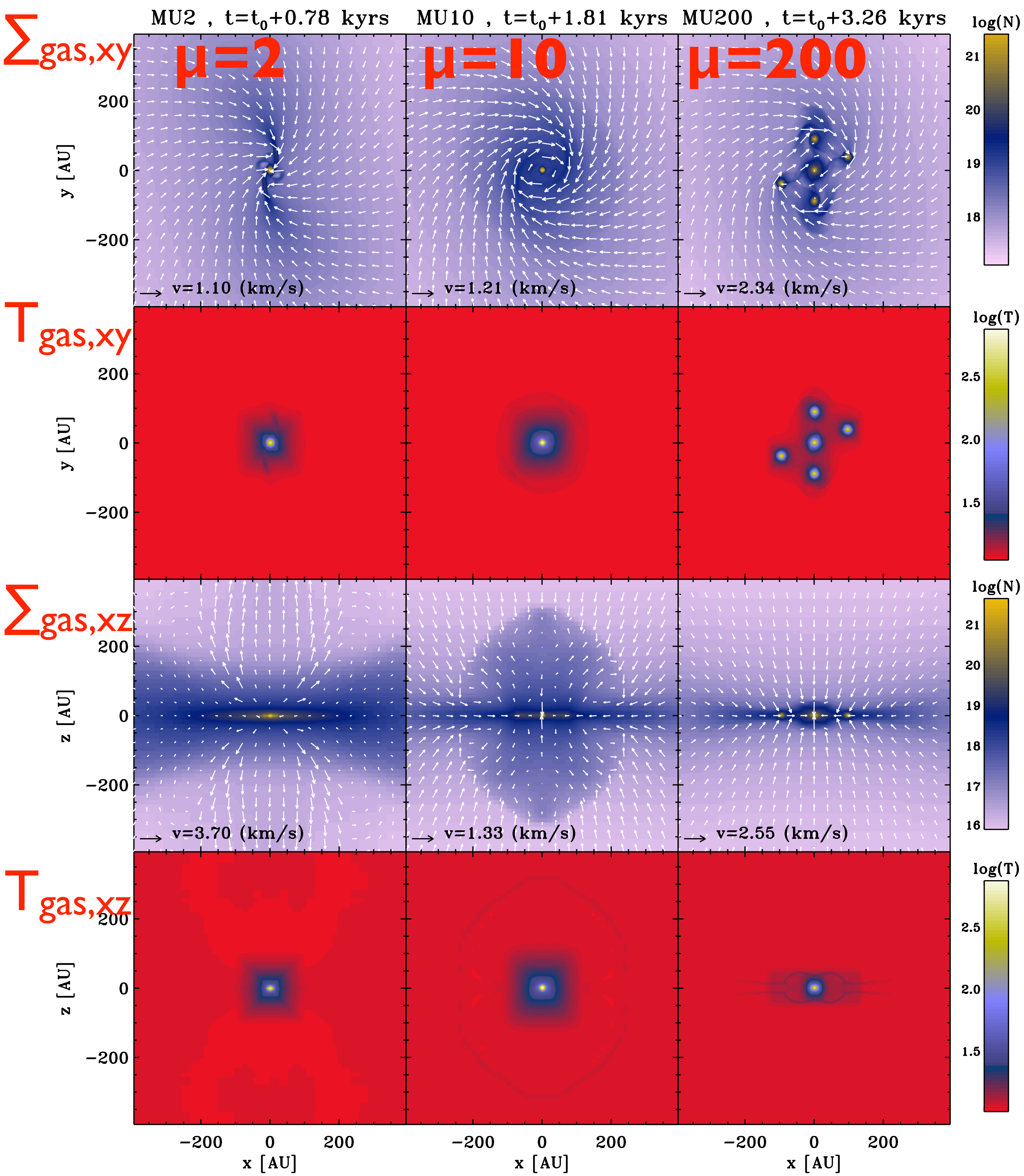


Gain in CPU time compared to DTU: factor 10



Commerçon et al. (2014)

Towards synthetic observations



- 3 representative cases
  - *MU2*: pseudo-disk + outflow
  - *MU10*: disk + pseudo-disk + outflow
  - *MU200*: disk + fragmentation

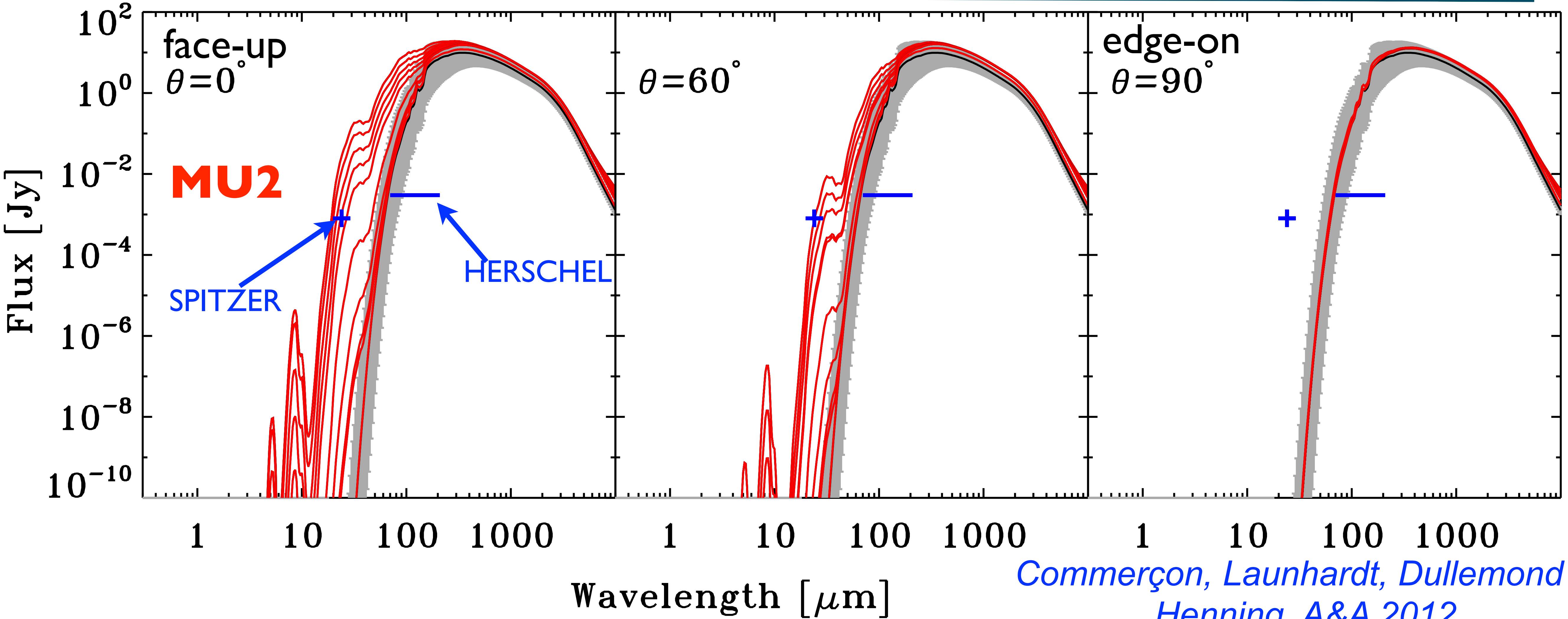
- First core lifetime:

MU2	*MU10*	*MU200*
1.2 kyr	3 kyr	> 4 kyr

- Images & SED computed with the radiative transfer code **RADMC-3D**, developed by C. Dullemond (ITA Heidelberg)
- $T_{dust} = T_{gas}$ (given by the RMHD calculations)

Commerçon, Launhardt, Dullemond & Henning, A&A 2012

SED - Do we see a first core signature?

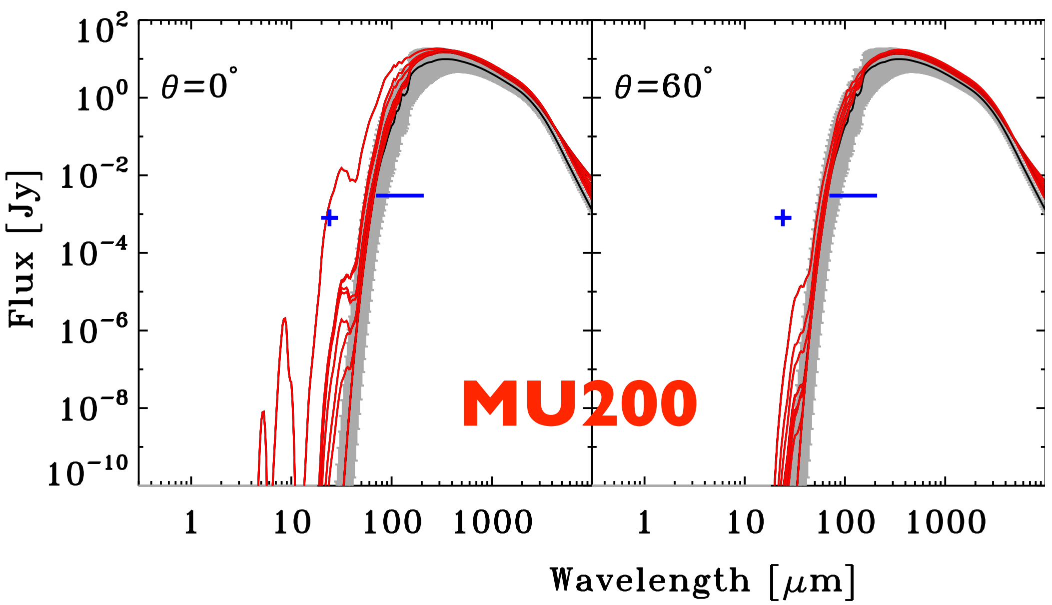


Commerçon, Launhardt, Dullemond & Henning, A&A 2012

- *Objects at 150 pc, 3000 AU x 3000 AU region*
- *Prestellar core = initial conditions (black line)*
- Emission in the FIR => **HERSCHEL, SPITZER**
- But similar SEDs in the MU200 model, i.e. **with a disk**!

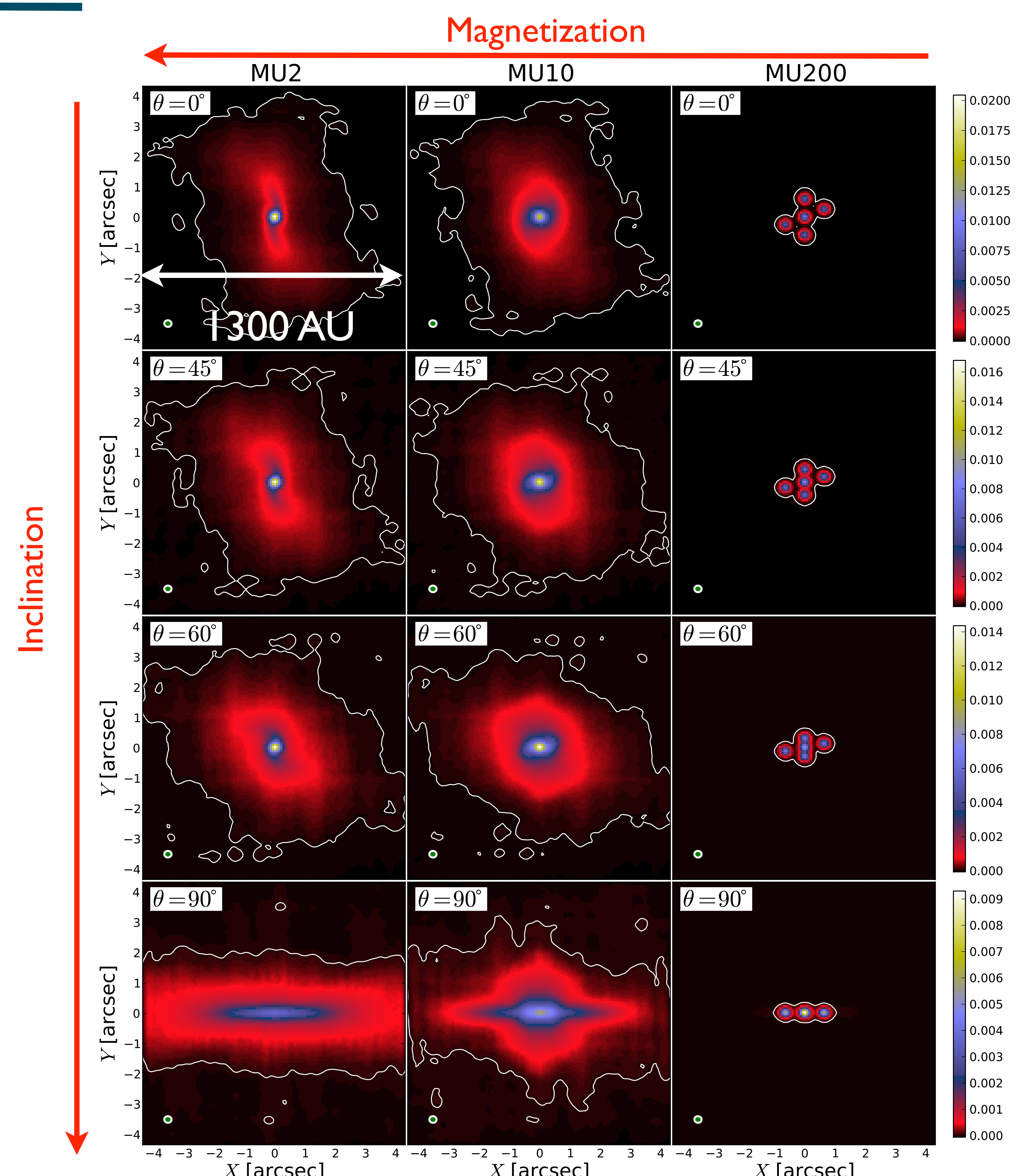
=> Issues in SED-fitting models for early Class 0?

Help to select first core candidates & to distinguish starless cores and first cores



Synthetic ALMA dust emission maps

ALMA Band 3 Config 20 @150 pc



Commerçon, Levrier et al. A&A, 2012

Extension to multigroup FLD

$$
\begin{aligned}
\partial_t \rho + \nabla \cdot [\rho \mathbf{u}] &= 0 \\
\partial_t (\rho \mathbf{u}) + \nabla \cdot [\rho \mathbf{u} \otimes \mathbf{u} + P\mathbb{I}] &= -\sum_{g=1}^{Ng} \lambda_g \nabla E_g \\
\partial_t E_{\mathrm{T}} + \nabla \cdot [\mathbf{u}(E_{\mathrm{T}} + P)] &= \sum_{g=1}^{Ng} \Bigg[-\mathbb{P}_g : \nabla \mathbf{u} - \lambda_g \mathbf{u} \cdot \nabla E_g \\
&\quad + \nabla \cdot \left(\frac{c\lambda_g}{\rho \kappa_{\mathrm{R}g}} \nabla E_g \right) \Bigg] \\
\partial_t E_g + \nabla \cdot [\mathbf{u} E_g] &= -\mathbb{P}_g : \nabla \mathbf{u} + \nabla \cdot \left(\frac{c\lambda_g}{\rho \kappa_{\mathrm{R}g}} \nabla E_g \right) \\
&\quad + \kappa_{\mathrm{P}g} \rho c \left(\Theta_g(T) - E_g \right) \\
&\quad + \nabla \mathbf{u} : \int_{\nu_{g-1/2}}^{\nu_{g+1/2}} \partial_\nu (\nu \mathbb{P}_\nu) d\nu
\end{aligned}
$$

✓ same operator split as in the previous grey model
+ one term of advection in frequency space (Doppler effect)

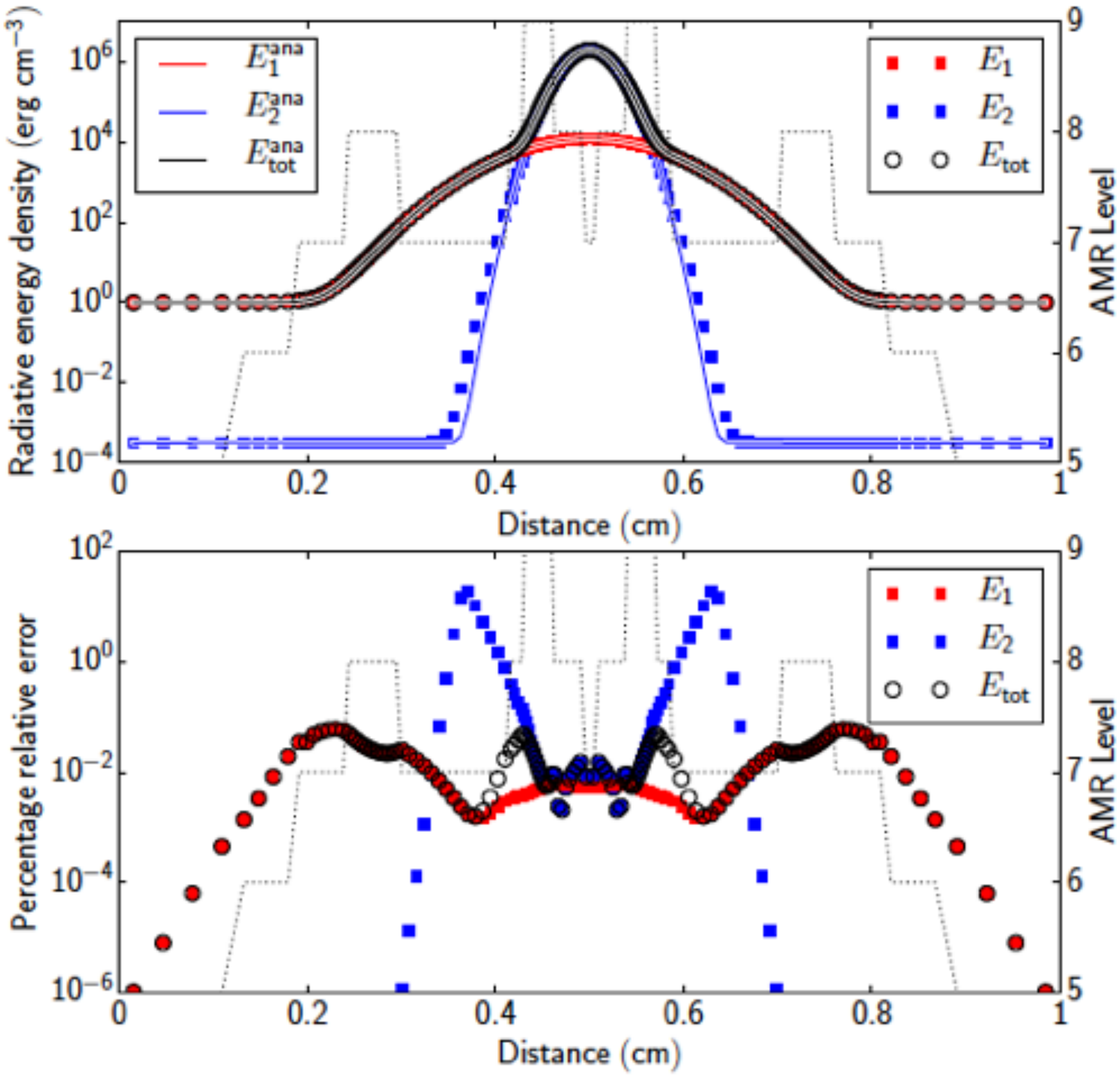
- N_g+1 coupled equations
- Linearized source term
- Non-symmetric matrix to invert
- BiCGStab iterative method (x2 more operations compared to CG)
- in the grey approx., it reduces to CG

González et al. (2015)

$$
T_i^{n+1} = \frac{C_{\mathrm{v}i}^n T_i^n - \sum_g \kappa_{\mathrm{P}g,i}^n \rho_i^n c\Delta t \left(\Theta_g(T_i^n) - T_i^n \Theta_g'(T_i^n) - E_{g,i}^{n+1} \right)}{C_{\mathrm{v}i}^n + \sum_g \kappa_{\mathrm{P}g,i}^n \rho_i^n c\Delta t \Theta_g'(T_i^n)}
$$

$$
\begin{aligned}
& E_{g,i}^{n+1} \left[1 + \kappa_{\mathrm{P}g,i}^n \rho_i^n c\Delta t + \frac{c\Delta t}{V_i} \left(\frac{\lambda_g}{\rho^n \kappa_{\mathrm{R}g}^n} \frac{S}{\Delta x} \right)_{i-1/2} + \frac{c\Delta t}{V_i} \left(\frac{\lambda_g}{\rho^n \kappa_{\mathrm{R}g}^n} \frac{S}{\Delta x} \right)_{i+1/2} \right] \\
& - \frac{c\Delta t}{V_i} \left(\frac{\lambda_g}{\rho^n \kappa_{\mathrm{R}g}^n} \frac{S}{\Delta x} \right)_{i-1/2} E_{g,i-1}^{n+1} - \frac{c\Delta t}{V_i} \left(\frac{\lambda_g}{\rho^n \kappa_{\mathrm{R}g}^n} \frac{S}{\Delta x} \right)_{i+1/2} E_{g,i+1}^{n+1} \\
& - \kappa_{\mathrm{P}g,i}^n \rho_i^n c\Delta t \Theta_g'(T_i^n) \sum_\alpha \frac{\kappa_{\mathrm{P}\alpha,i}^n \rho_i^n c\Delta t}{C_{\mathrm{v}i}^n + \sum_\beta \kappa_{\mathrm{P}\beta,i}^n \rho_i^n c\Delta t \Theta_\beta'(T_i^n)} E_{\alpha,i}^{n+1} \\
& \qquad = E_{g,i}^n + \kappa_{\mathrm{P}g,i}^n \rho_i^n c\Delta t \left(\Theta_g(T_i^n) - T_i^n \Theta_g'(T_i^n) \right) \\
& + \kappa_{\mathrm{P}g,i}^n \rho_i^n c\Delta t \Theta_g'(T_i^n) \frac{C_{\mathrm{v}i}^n T_i^n - \sum_\alpha \kappa_{\mathrm{P}\alpha,i}^n \rho_i^n c\Delta t \left(\Theta_\alpha(T_i^n) - T_i^n \Theta_\alpha'(T_i^n) \right)}{C_{\mathrm{v}i}^n + \sum_\alpha \kappa_{\mathrm{P}\alpha,i}^n \rho_i^n c\Delta t \Theta_\alpha'(T_i^n)} .
\end{aligned}
$$

1D Dirac diffusion with 2 energy groups

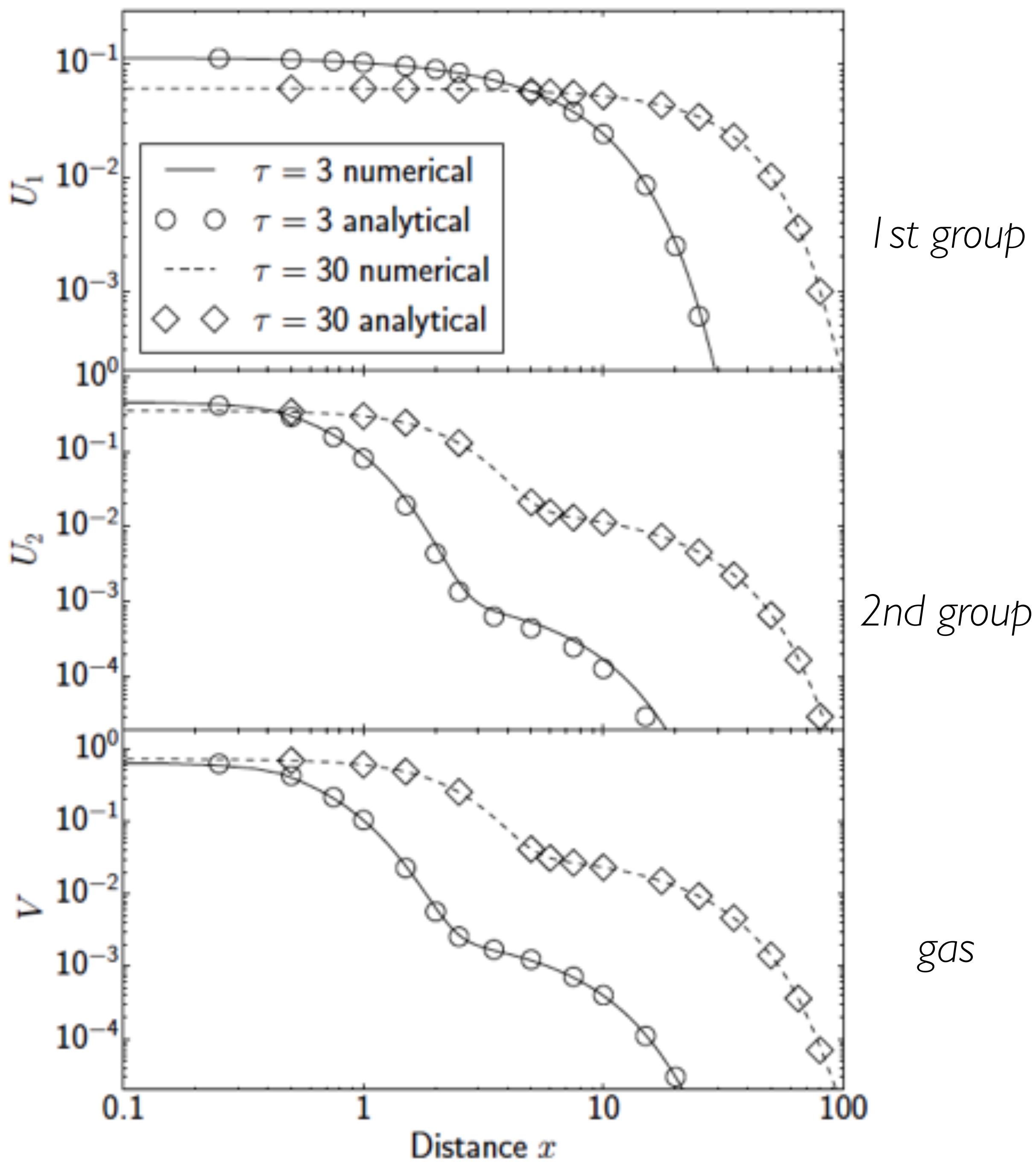


$$\partial_t E_1 - \nabla\left(\frac{c}{3\rho\kappa_{R1}}\nabla E_1\right) = 0$$
$$\partial_t E_2 - \nabla\left(\frac{c}{3\rho\kappa_{R2}}\nabla E_2\right) = 0.$$

$$\kappa_{R1} = 1; \kappa_{R2} = 10$$

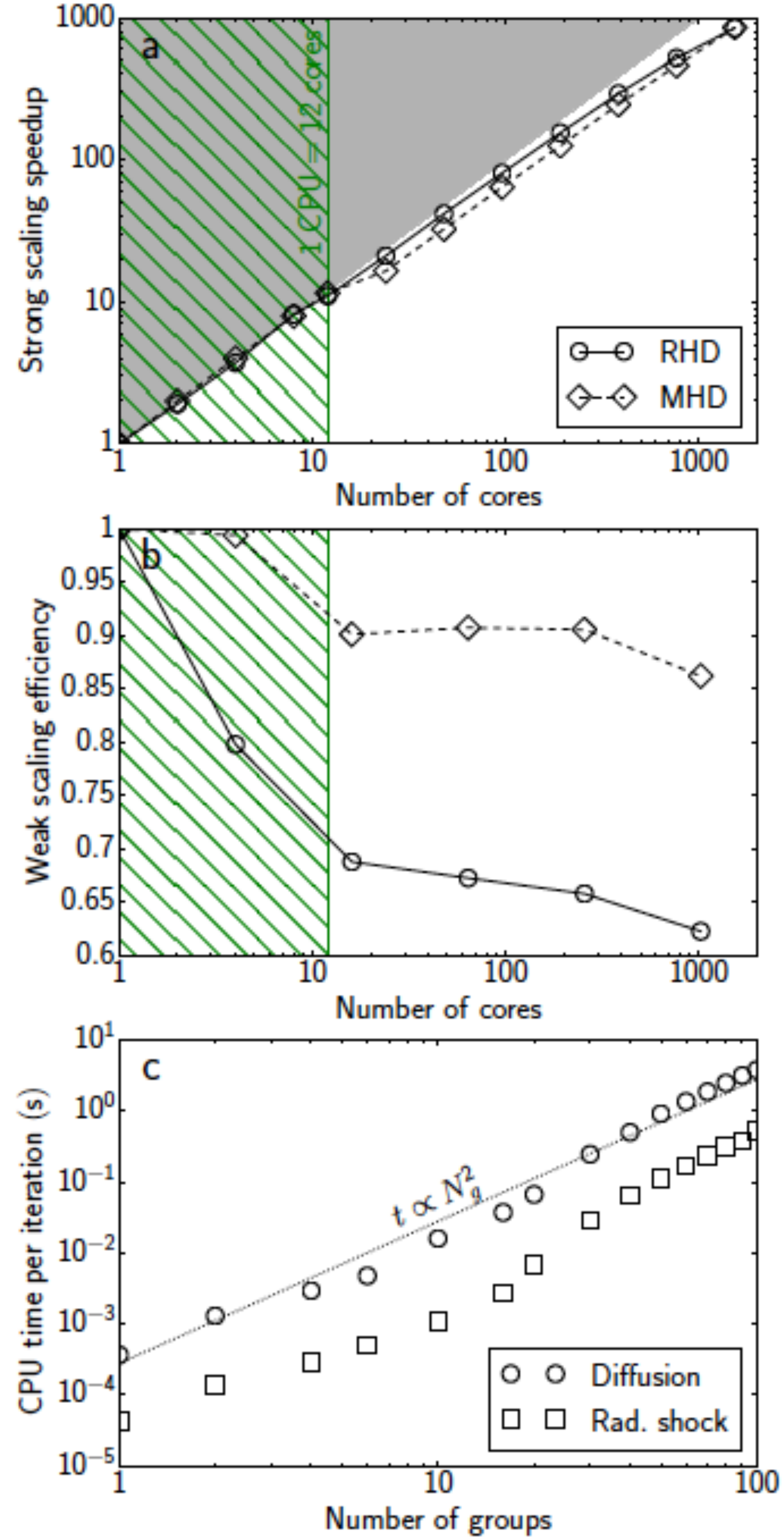
González et al. (2015)

Radiative tests (no hydro)



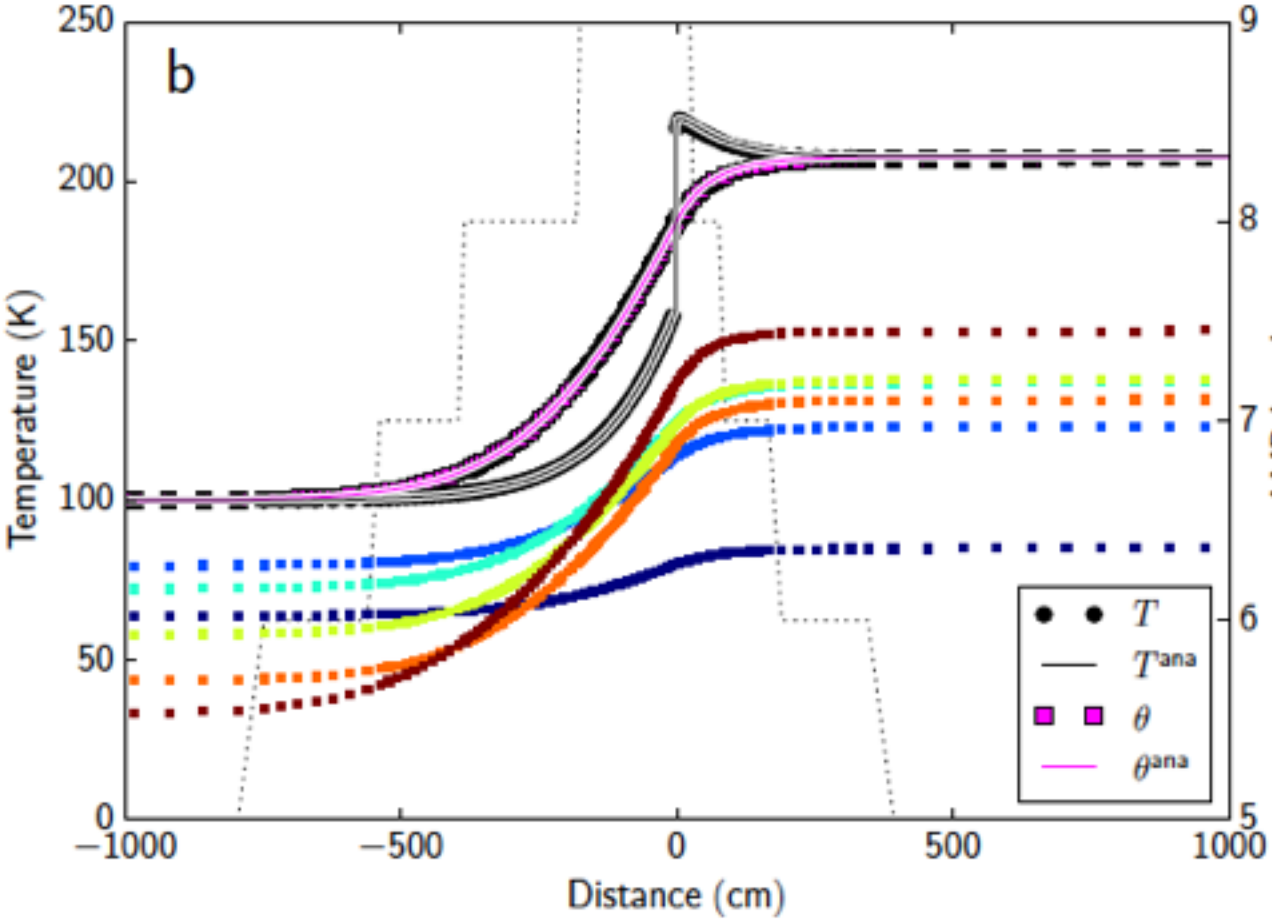
Su-Olson test

González et al. (2015)

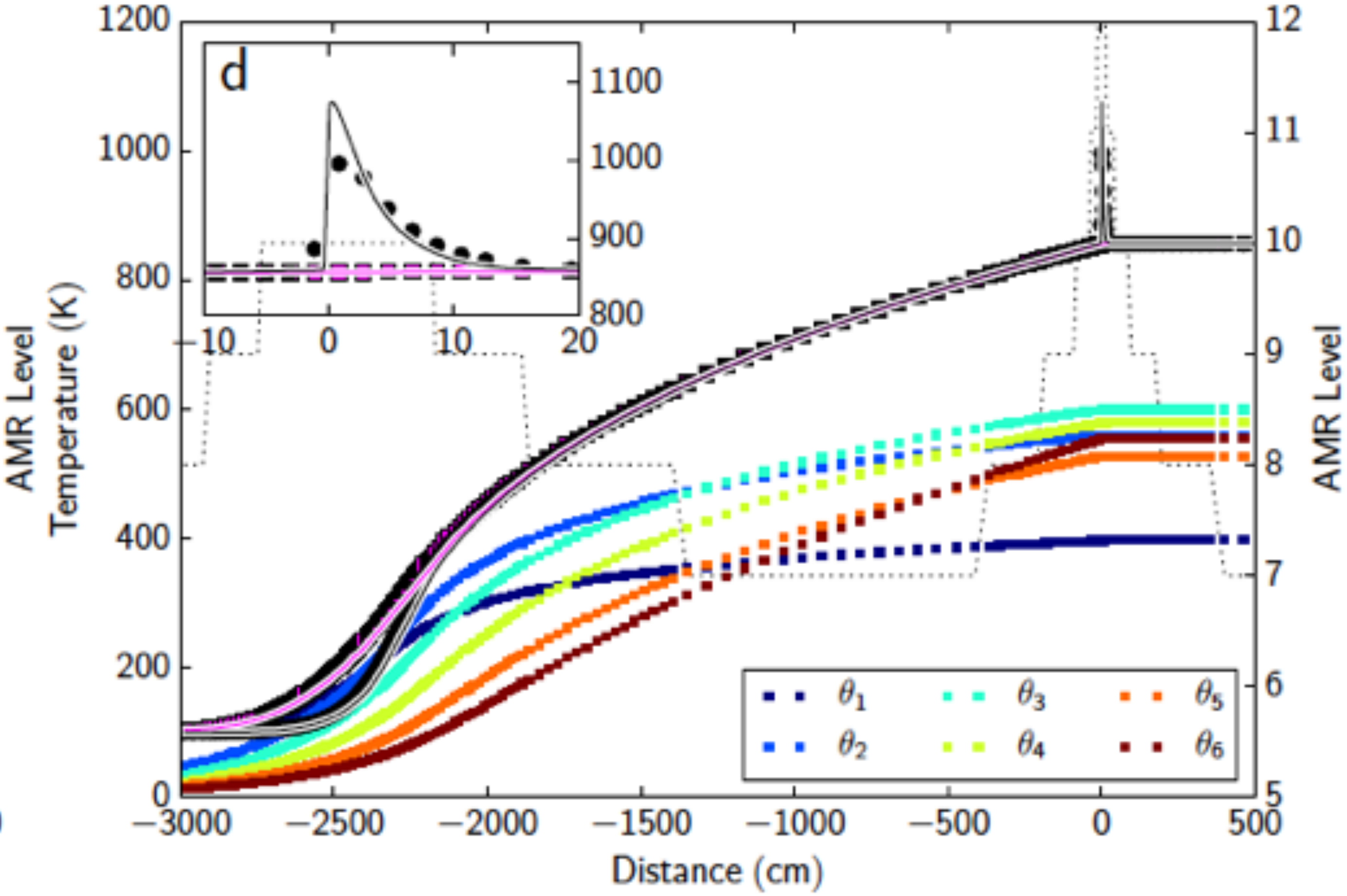


Scalability tests

Radiative shocks



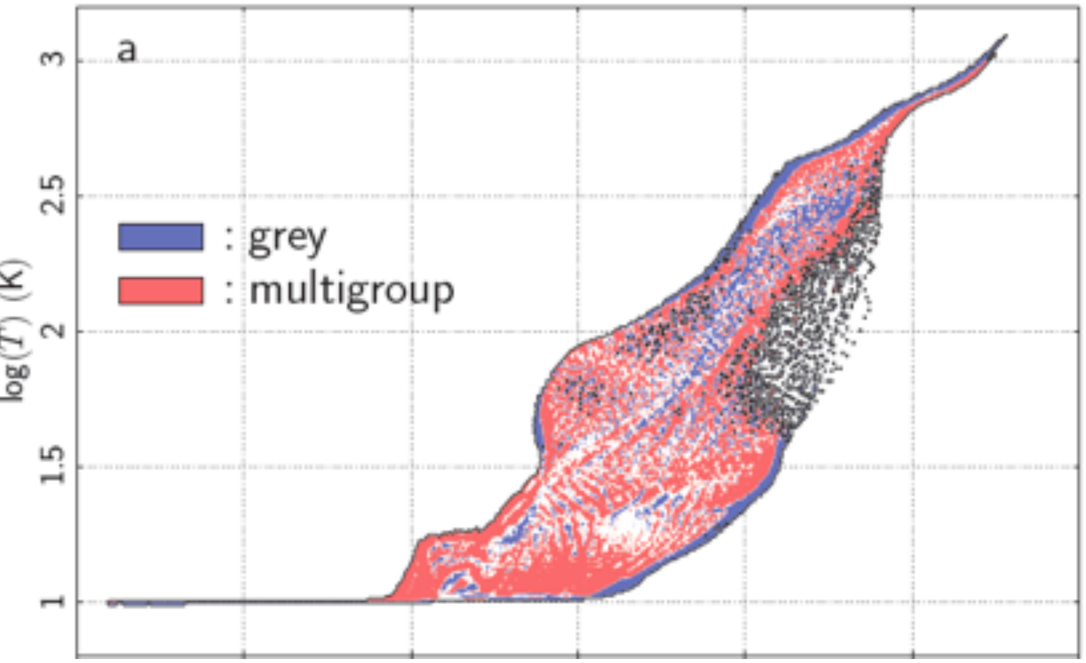
Subcritical shock



Supercritical shock

González et al. (2015)

Application to star formation: protostellar collapse



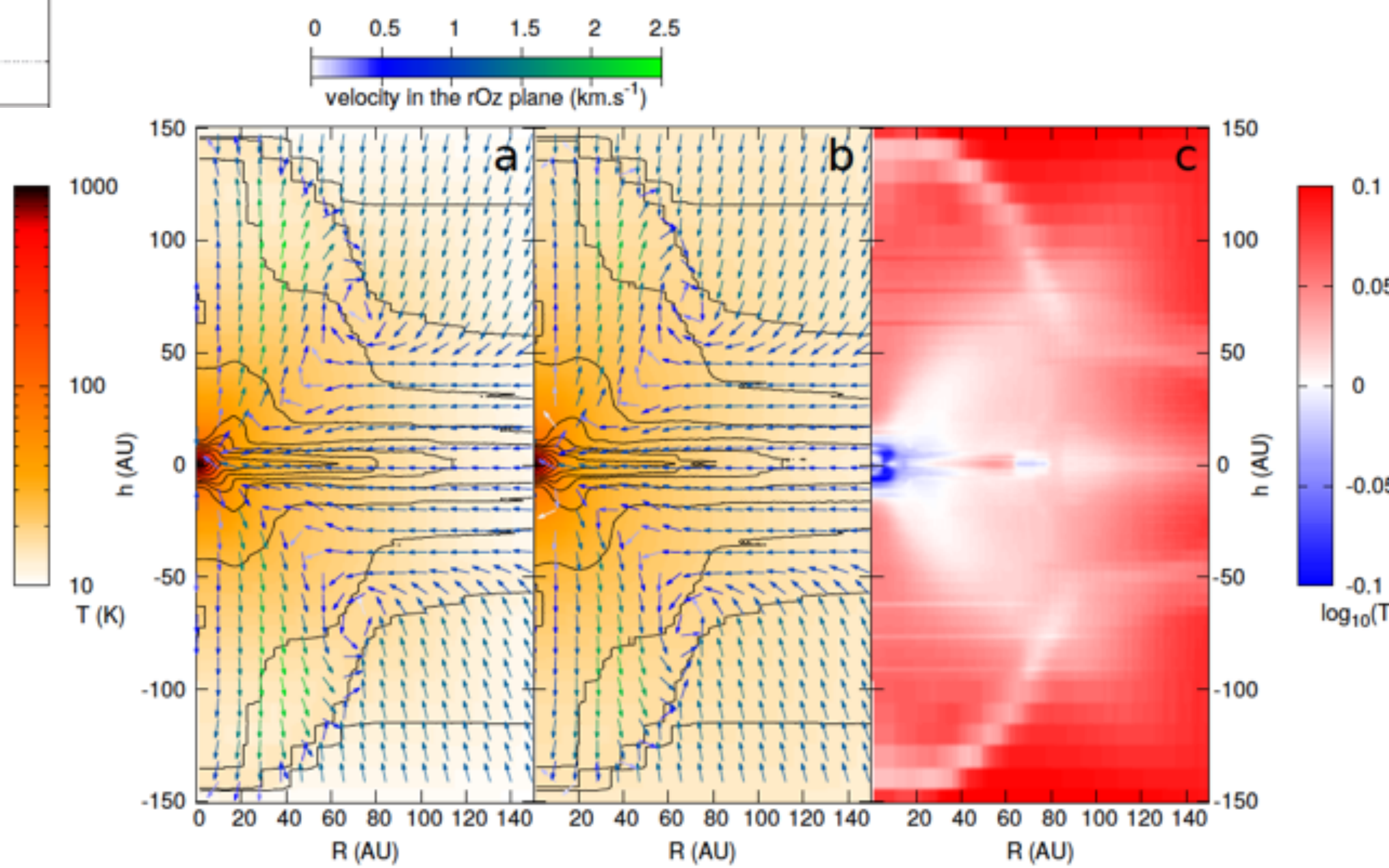
Temperature-density distribution

1 M⊙ magnetised dense core (Boss & Bodenheimer test case)

- 2 simulations : grey + multi group with 20 frequency bins
- ideal MHD (**μ**=5)

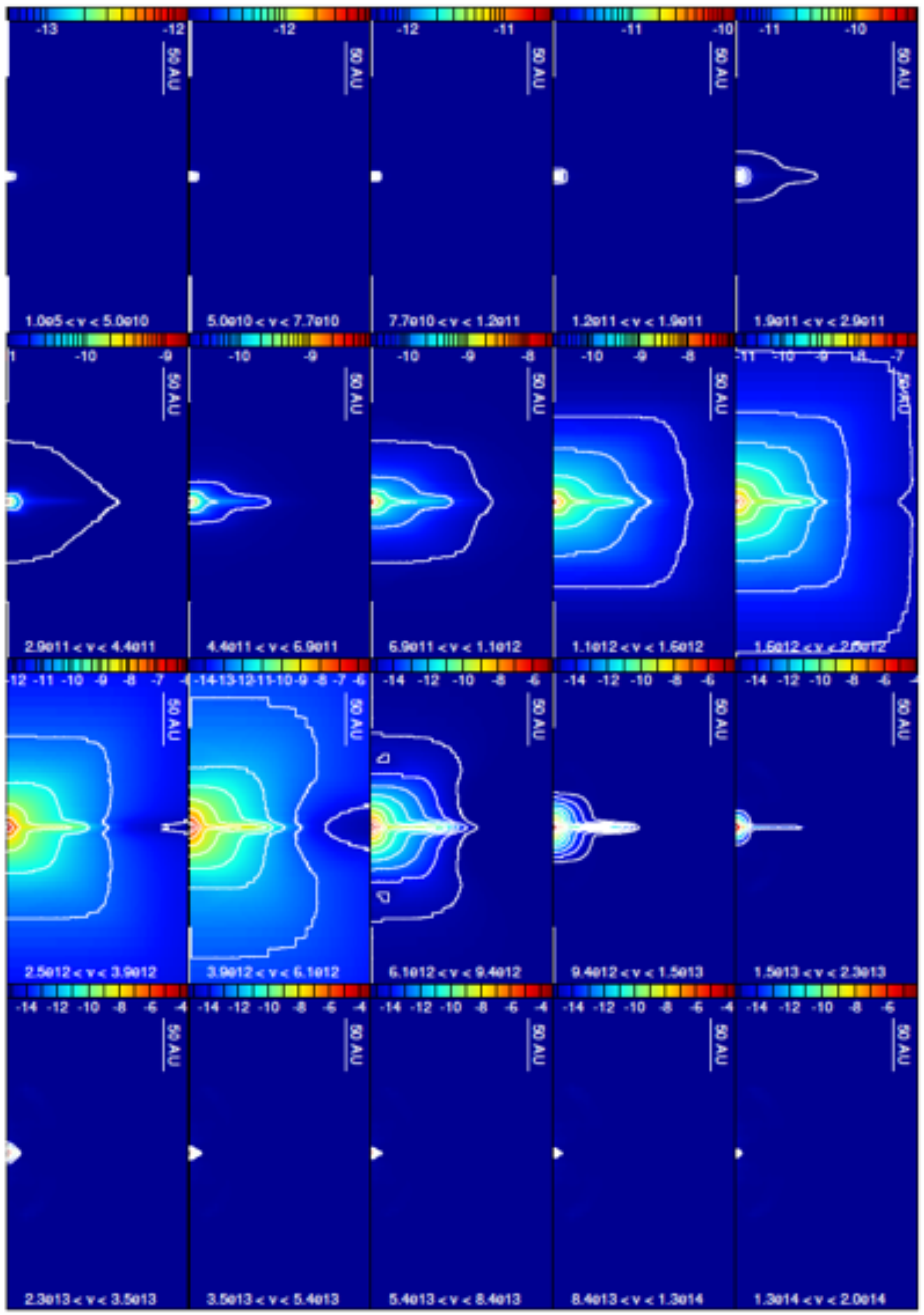
Gas colder within the first Larson core, but warmer in the envelop and in the outflow
=> same effects for more massive core?

González et al. (2015)

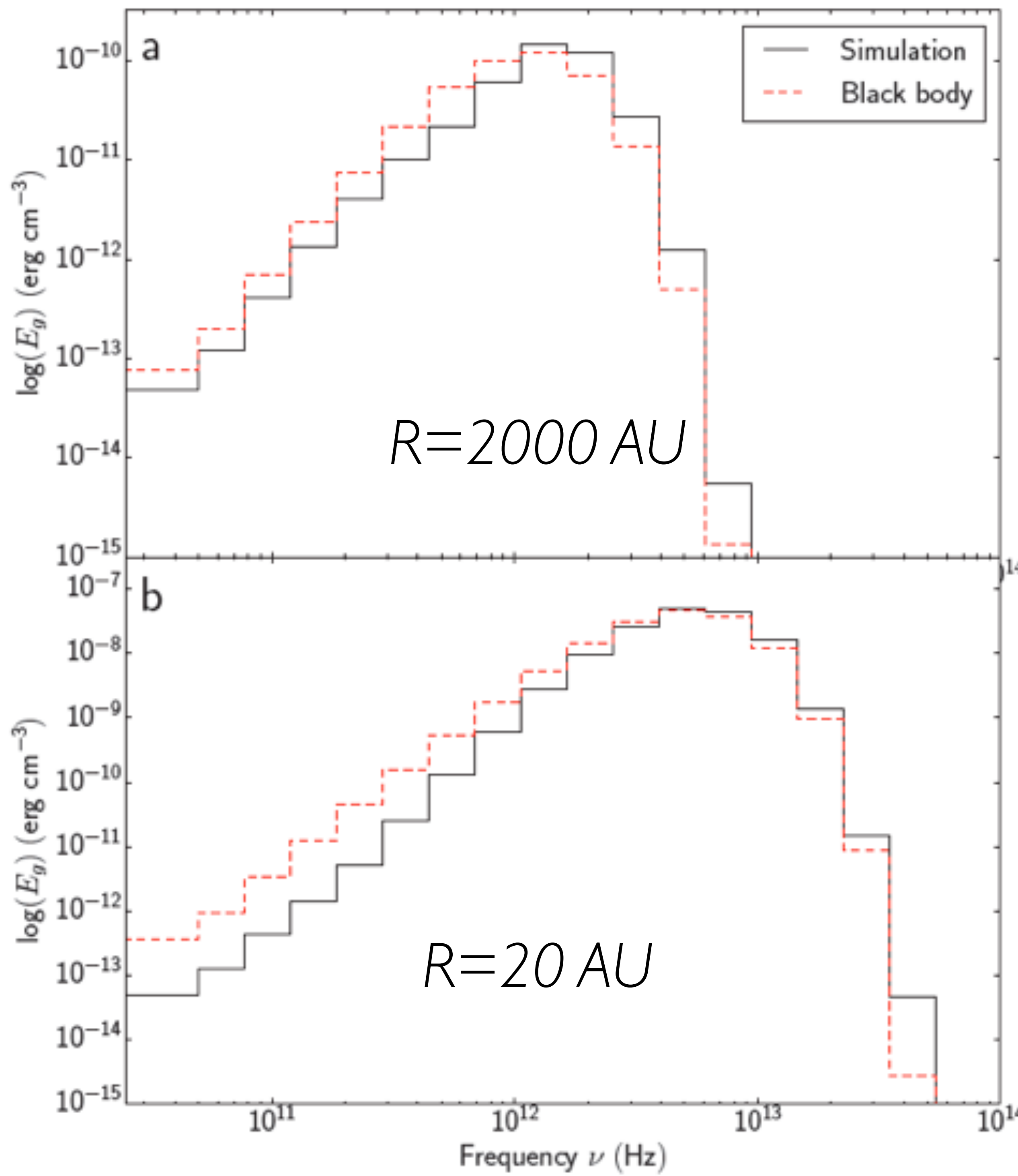


Grey *20 bins*

Link with observations



Frequency channel maps

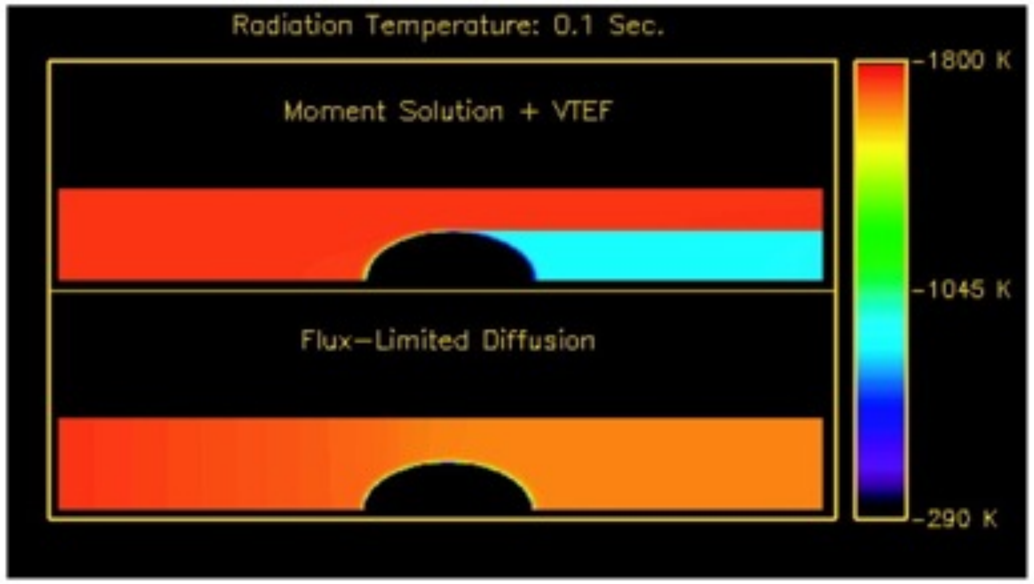


Spectral energy distribution

González et al. (2015)

Beyond FLD

- **FLD has limitations…**
  - isotropy
  - streaming limit



Shadow test (Hayes & Normann 2003)

- **M1** method (e.g., HERACLES code, *González et al. 2007,* RAMSES_RT, *Rosdahl et al. 2013*)
- **VET (Variable Eddington Tensor)** method (e.g., ZEUS code, *Stone et al. 1992,* ATHENA code, *Davis et al. 2012,* OTVET,Gnedin & Abel 2001)
- **Irradiation + FLD** (e.g., PLUTO code, *Kuiper et al. 2010, Flock et al. 2013,* FLASH, *Klassen et al. 2016*)
- **Monte Carlo** RHD (e.g., TORUS code, *Harries 2015*)

Non-exhaustive list…

Cosmic rays hydrodynamics on grids

Benoît Commerçon
Centre de Recherche Astrophysique de Lyon
Yohan Dubois (IAP Paris)

Dubois & Commerçon (2016)

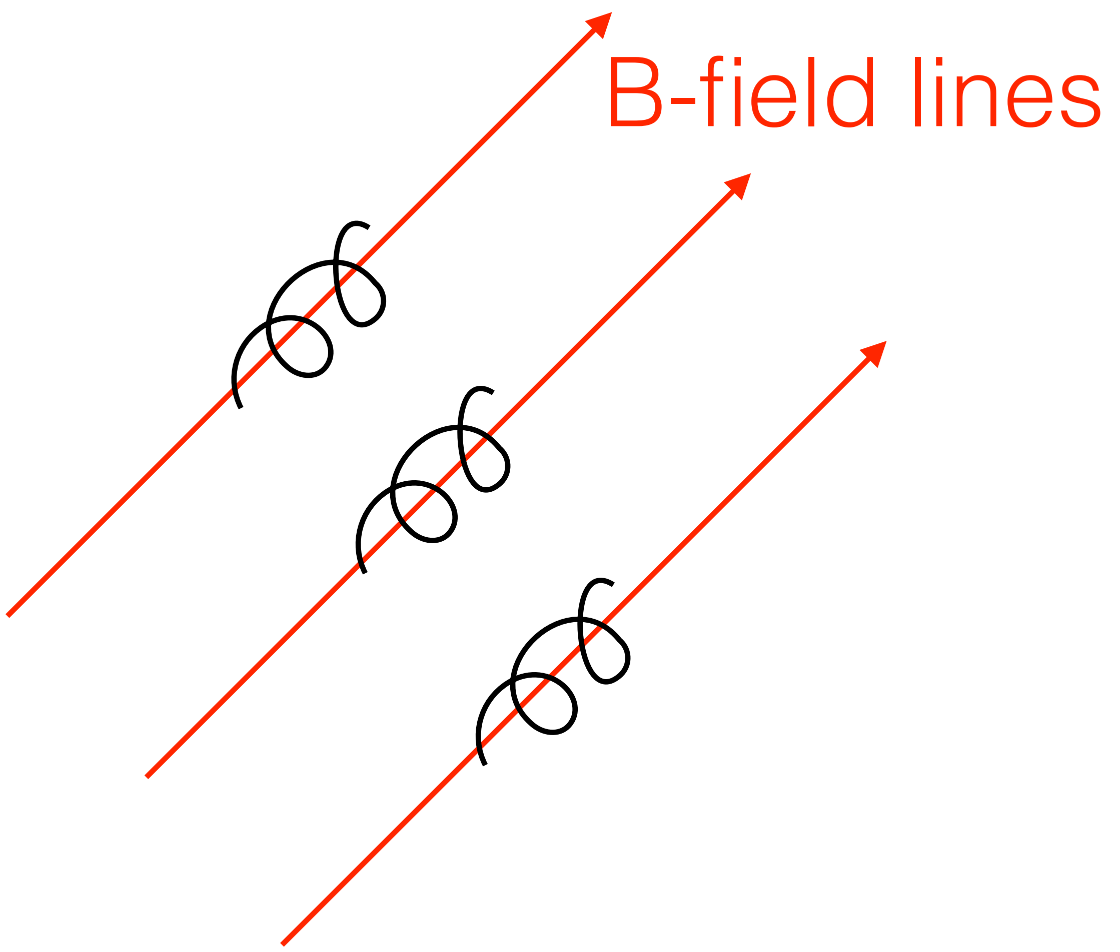
Why conduction of heat?

- Kinetic energy is converted into thermal energy at shocks.
- The conduction of heat can spread the shock and reheat regions that would have remained cold otherwise.
- Heat is primarily conducted by electrons (lighter population).
    - ➡If the coupling timescale of ion and electron temperatures is larger than the diffusion timescale (or than the eddy-turnover timescale): ion and electron temperatures differ.

Courtesy Y. Dubois

Why anisotropic?

- Magnetic field is present everywhere in astrophysics.
- Charged particles diffuse along magnetic field lines.



Courtesy Y. Dubois

Anisotropic diffusion in RAMSES

- Fluid of gas + cosmic ray with total energy $e = e_{\rm int} + \rho u^2/2 + B^2/2 + e_{\rm cr}$

mass equation
$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \boldsymbol{u}) = 0\,,$$

momentum equation
$$\frac{\partial \rho \boldsymbol{u}}{\partial t} + \nabla.\left(\rho \boldsymbol{u}\boldsymbol{u} + p_{\rm tot} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi}\right) = 0\,,$$

total energy equation
$$\frac{\partial e}{\partial t} + \nabla.\left((e + p_{\rm tot})\boldsymbol{u} - \frac{\boldsymbol{B}(\boldsymbol{B}.\boldsymbol{u})}{4\pi}\right) = -\nabla.\boldsymbol{F}_{\rm cond} - \nabla.\boldsymbol{F}_{\rm CR}\,,$$

induction equation
$$\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = 0\,,$$

electron energy equation
$$\frac{\partial e_{\rm E}}{\partial t} + \nabla.(e_{\rm E}\boldsymbol{u}) = -p_{\rm E}\nabla.\boldsymbol{u} - \nabla.\boldsymbol{F}_{\rm cond} + \mathcal{H}_{\rm EI}$$

CR energy equation
$$\frac{\partial e_{\rm cr}}{\partial t} + \nabla.(e_{\rm cr}\boldsymbol{u}) = -p_{\rm cr}\nabla.\boldsymbol{u} - \nabla.\boldsymbol{F}_{\rm CR}\,,$$

Conservative terms

Source terms

- modified sound speed $\tilde{c}_{\rm s} = \sqrt{c_{\rm s} + \gamma_{\rm cr}(\gamma_{\rm cr} - 1)e_{\rm cr}}$
- anisotropic diffusion tensor…

Implicit integration of CR diffusion equation

$$\frac{\partial e_{\rm cr}}{\partial t} = \nabla \cdot (D_{\|}\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \cdot \nabla e_{\rm cr}) + \nabla \cdot [D_{\perp}(\boldsymbol{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}})\nabla e_{\rm cr}]$$

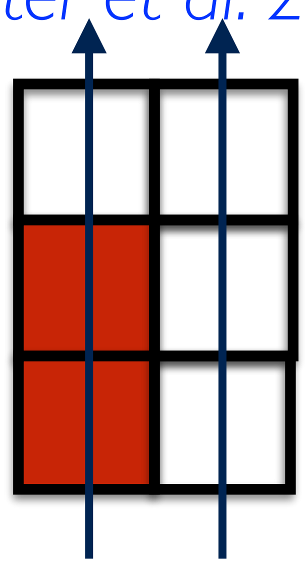
- CFL condition is $\Delta t_{\rm exp} < \frac{\Delta x^2}{2D_{\|}}$ $\qquad D_{\perp} = 0.01 D_{\|}$

✓ *But* implicit scheme is unconditionally stable

➡ **Implicit discretization using a centred symmetric scheme** (*Günter et al. 2005*):

$$e_{i,j}^{n+1} = f(e_{i,j}^{n}, e_{i-1,j}^{n+1}, e_{i+1,j}^{n+1}, e_{i,j-1}^{n+1}, e_{i,j+1}^{n+1})$$

+ easy to solve using conjugate gradient
- does not preserve monotonicity (negative values)



➡ **implicit adaptive time-step** (*Commerçon et al. 2014*)

Implicit integration of CR diffusion equation

$$\frac{\partial e_{\rm cr}}{\partial t} = \nabla \cdot (D_{\parallel} \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \cdot \nabla e_{\rm cr}) + \nabla \cdot [D_{\perp}(\boldsymbol{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}})\nabla e_{\rm cr}]$$

- CFL condition is $\Delta t_{\rm exp} < \frac{\Delta x^2}{2D_{\parallel}}$ $\quad D_{\perp} = 0.01 D_{\parallel}$

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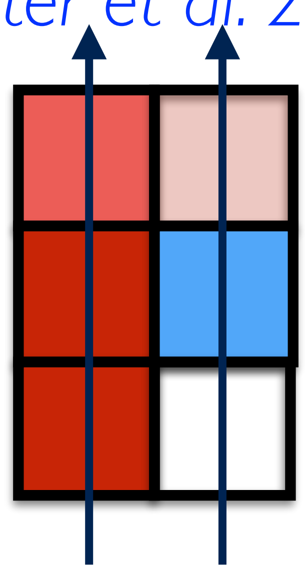
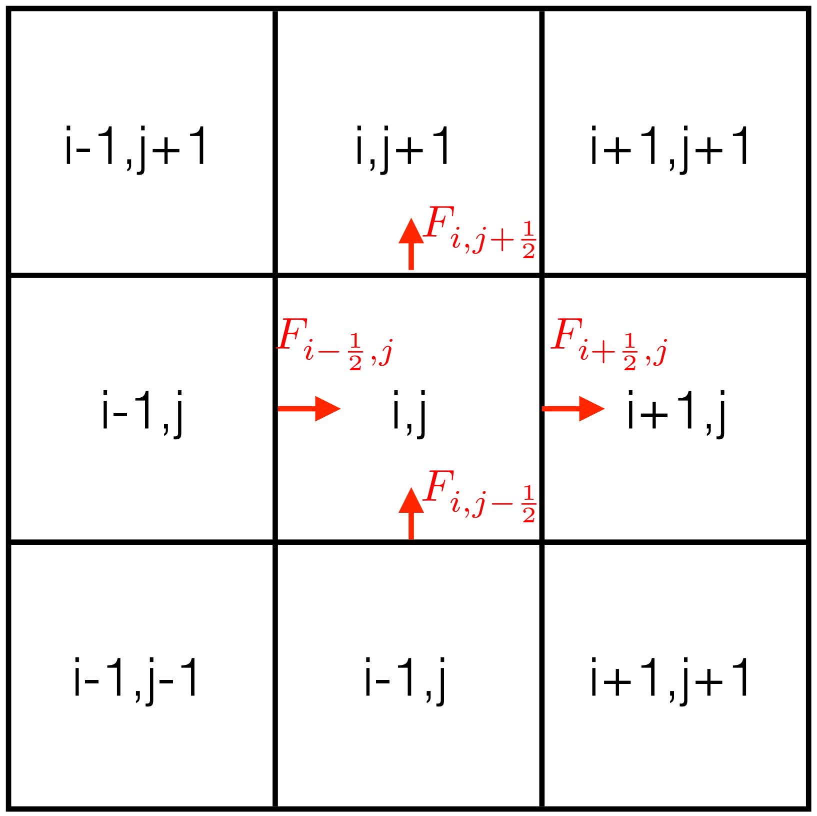


+ easy to solve using conjugate gradient
- does not preserve monotonicity (negative values)

➡ **implicit adaptive time-step** (*Commerçon et al. 2014*)

Anisotropic conduction in 2D

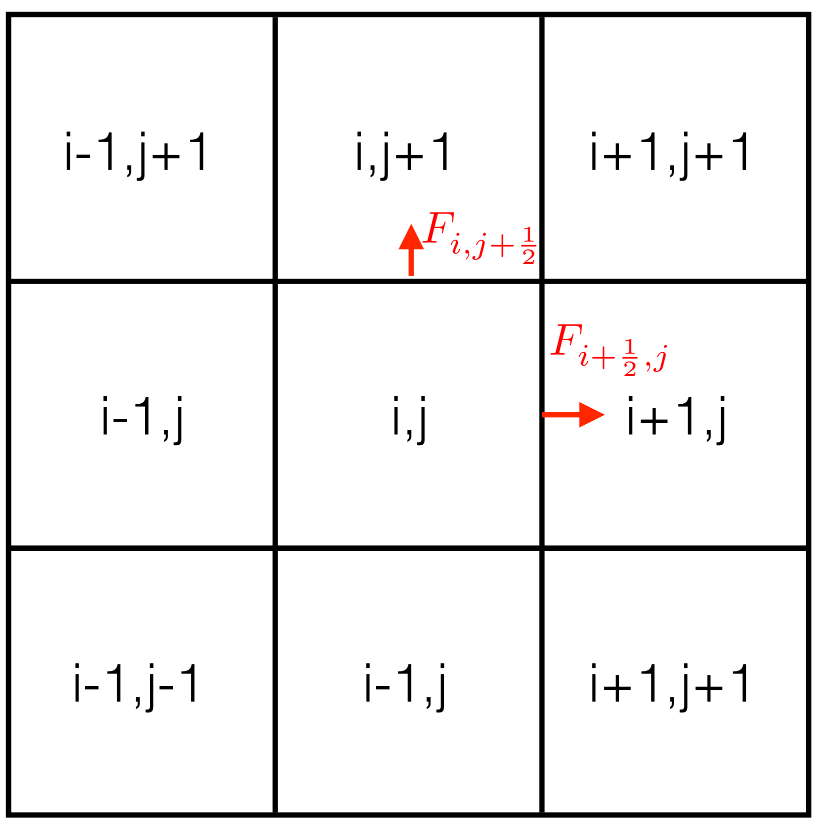
$$e^{n+1}_{i,j} + \Delta t \frac{F^{n+1}_{i+\frac{1}{2},j} + F^{n+1}_{i,j+\frac{1}{2}} - F^{n+1}_{i-\frac{1}{2},j} - F^{n+1}_{i,j-\frac{1}{2}}}{\Delta x} = e^{n}_{i,j}$$



Courtesy Y. Dubois

Anisotropic conduction in 2D

$$e^{n+1}_{i,j} + \Delta t \frac{F^{n+1}_{i+\frac{1}{2},j} + F^{n+1}_{i,j+\frac{1}{2}} - F^{n+1}_{i-\frac{1}{2},j} - F^{n+1}_{i,j-\frac{1}{2}}}{\Delta x} = e^n_{i,j}$$



$$F^{\rm ani}_{i+\frac{1}{2},j} = \frac{F^{\rm ani}_{i+\frac{1}{2},j-\frac{1}{2}} + F^{\rm ani}_{i+\frac{1}{2},j+\frac{1}{2}}}{2}$$

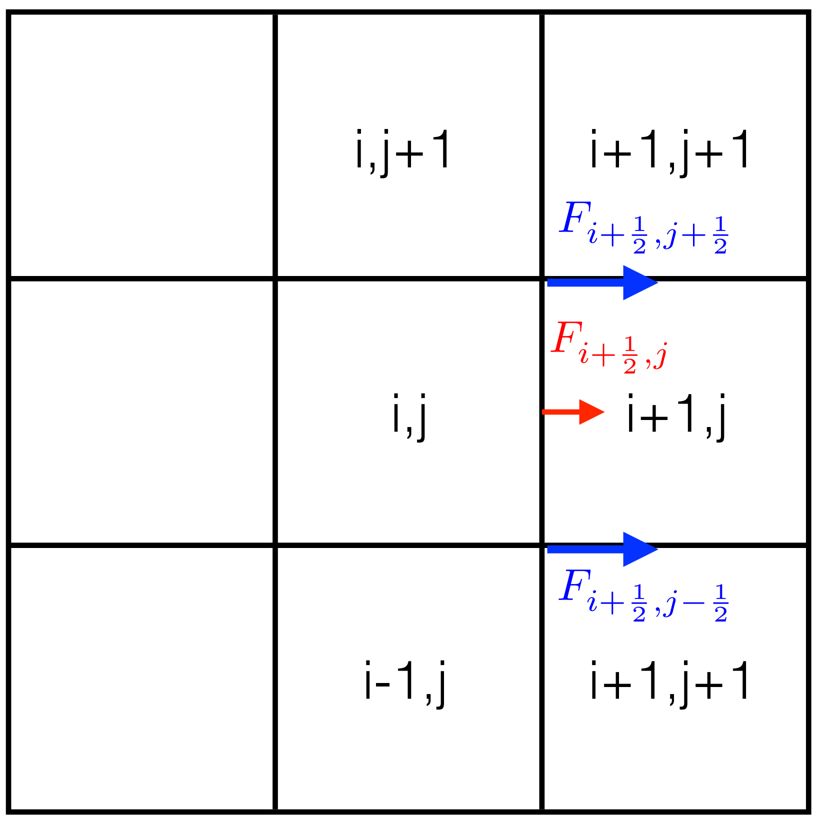
$$F^{\rm ani}_{i,j+\frac{1}{2}} = \frac{F^{\rm ani}_{i-\frac{1}{2},j+\frac{1}{2}} + F^{\rm ani}_{i+\frac{1}{2},j+\frac{1}{2}}}{2}$$

Courtesy Y. Dubois

Centred symmetric scheme
(Günter et al, 2005)

Anisotropic conduction in 2D

$$e_{i,j}^{n+1} + \Delta t \frac{F_{i+\frac{1}{2},j}^{n+1} + F_{i,j+\frac{1}{2}}^{n+1} - F_{i-\frac{1}{2},j}^{n+1} - F_{i,j-\frac{1}{2}}^{n+1}}{\Delta x} = e_{i,j}^{n}$$



$$F^{ani}_{i+\frac{1}{2},j} = \frac{F^{ani}_{i+\frac{1}{2},j-\frac{1}{2}} + F^{ani}_{i+\frac{1}{2},j+\frac{1}{2}}}{2}$$

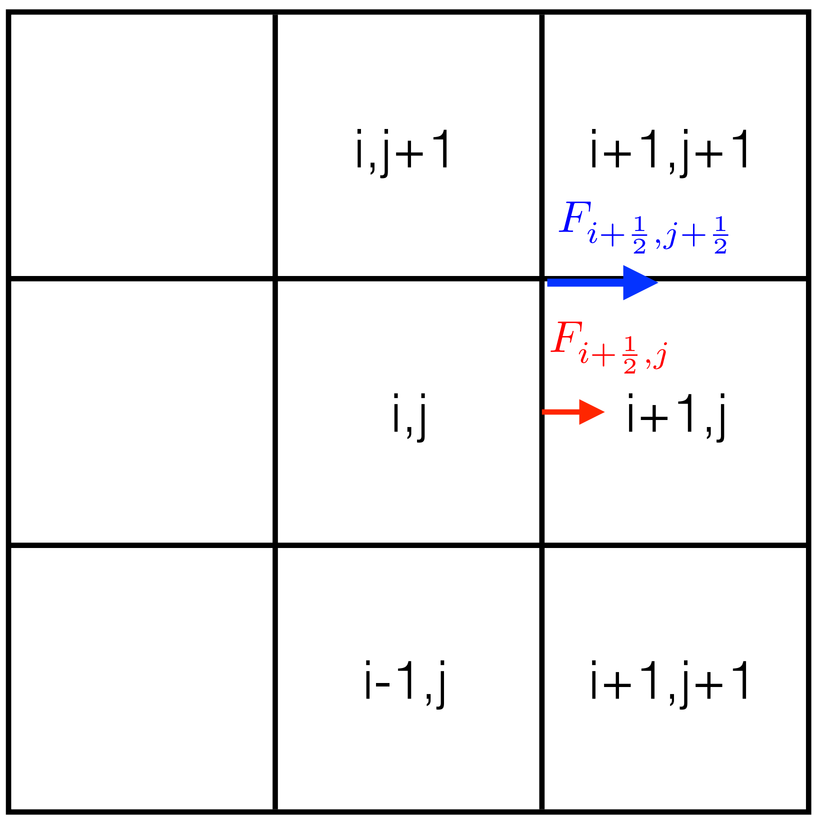
$$F^{ani}_{i,j+\frac{1}{2}} = \frac{F^{ani}_{i-\frac{1}{2},j+\frac{1}{2}} + F^{ani}_{i+\frac{1}{2},j+\frac{1}{2}}}{2}$$

Courtesy Y. Dubois

Centred symmetric scheme
(Günter et al, 2005)

Anisotropic conduction in 2D

$$e_{i,j}^{n+1} + \Delta t \frac{F_{i+\frac{1}{2},j}^{n+1} + F_{i,j+\frac{1}{2}}^{n+1} - F_{i-\frac{1}{2},j}^{n+1} - F_{i,j-\frac{1}{2}}^{n+1}}{\Delta x} = e_{i,j}^{n}$$



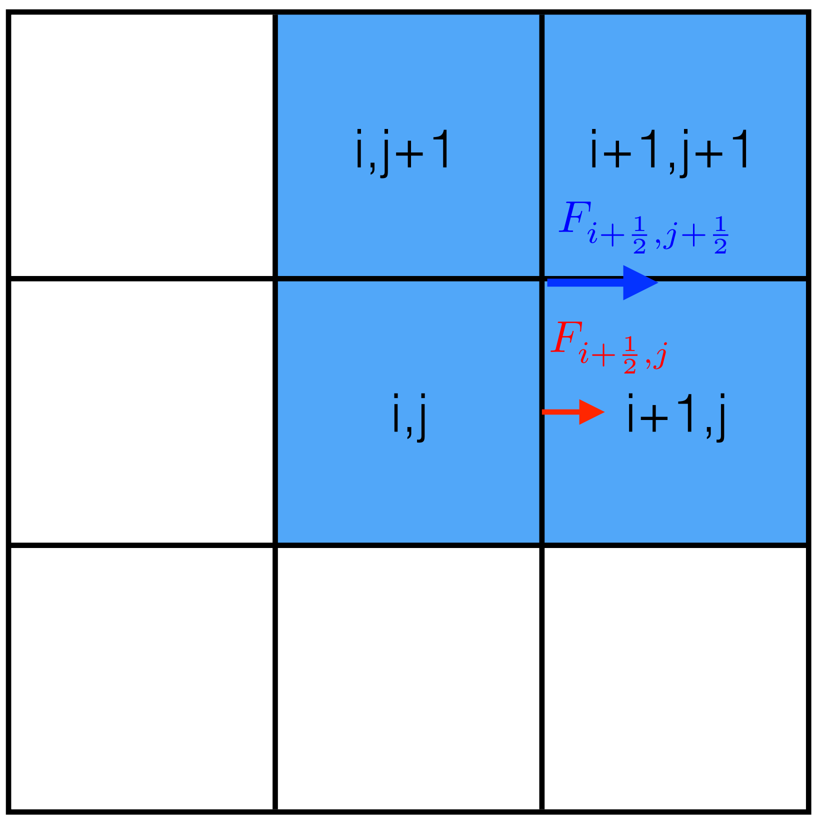
$$F_{i+\frac{1}{2},j+\frac{1}{2}}^{\mathrm{ani}} = \bar{\kappa}_{\parallel} \bar{b}_x \left(\bar{b}_x \frac{\partial \bar{T}}{\partial x} + \bar{b}_y \frac{\partial \bar{T}}{\partial y} \right)$$

Courtesy Y. Dubois

Centred symmetric scheme
(Günter et al, 2005)

Anisotropic conduction in 2D

$$e_{i,j}^{n+1} + \Delta t \frac{F_{i+\frac{1}{2},j}^{n+1} + F_{i,j+\frac{1}{2}}^{n+1} - F_{i-\frac{1}{2},j}^{n+1} - F_{i,j-\frac{1}{2}}^{n+1}}{\Delta x} = e_{i,j}^{n}$$



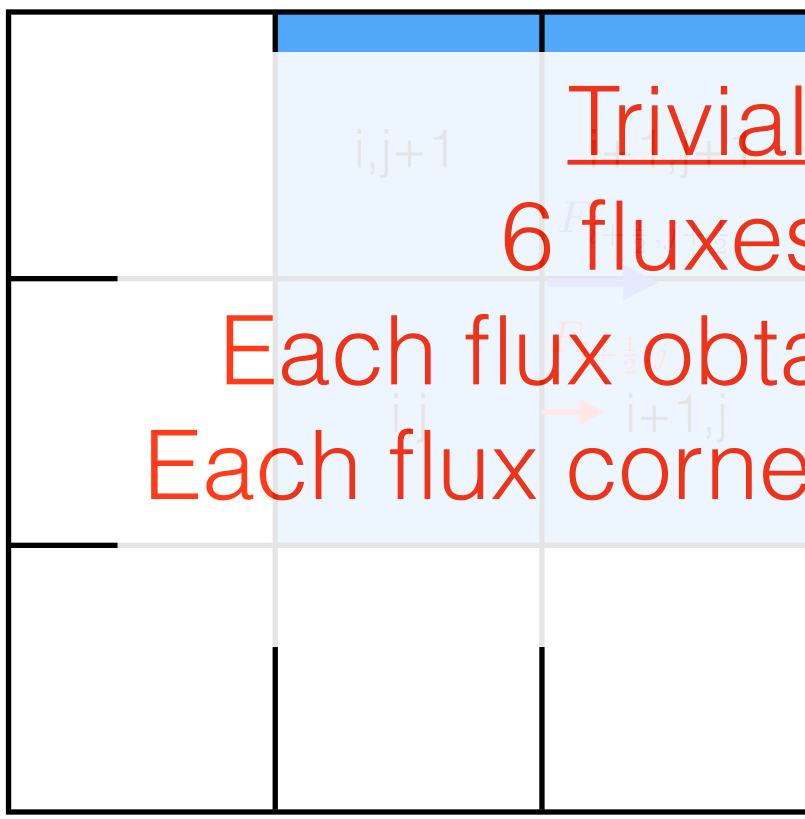
$$F_{i+\frac{1}{2},j+\frac{1}{2}}^{\mathrm{ani}} = \bar{\kappa}_{\|} \bar{b}_x \left(\bar{b}_x \frac{\bar{\partial} T}{\partial x} + \bar{b}_y \frac{\bar{\partial} T}{\partial y} \right)$$

$$\begin{aligned}
\bar{b}_x &= \frac{b_{x,i+\frac{1}{2},j}^{n} + b_{x,i+\frac{1}{2},j+1}^{n}}{2}, \\
\bar{b}_y &= \frac{b_{y,i,j+\frac{1}{2}}^{n} + b_{y,i+1,j+\frac{1}{2}}^{n}}{2}, \\
\frac{\bar{\partial} T}{\partial x} &= \frac{T_{i+1,j+1}^{n+1} + T_{i+1,j}^{n+1} - T_{i,j+1}^{n+1} - T_{i,j}^{n+1}}{2\Delta x}, \\
\frac{\bar{\partial} T}{\partial y} &= \frac{T_{i+1,j+1}^{n+1} + T_{i,j+1}^{n+1} - T_{i+1,j}^{n+1} - T_{i,j}^{n+1}}{2\Delta x}, \\
\bar{\kappa}_{\|} &= \frac{\kappa_{\|i,j}^{n} + \kappa_{\|i+1,j}^{n} + \kappa_{\|i,j+1}^{n} + \kappa_{\|i+1,j+1}^{n}}{4}.
\end{aligned}$$

Centred symmetric scheme
(Günter et al, 2005)

Anisotropic conduction in 2D

$$e_{i,j}^{n+1} + \Delta t \frac{F_{i+\frac{1}{2},j}^{n+1} + F_{i,j+\frac{1}{2}}^{n+1} - F_{i-\frac{1}{2},j}^{n+1} - F_{i,j-\frac{1}{2}}^{n+1}}{\Delta x} = e_{i,j}^{n}$$



$$F^{\rm ani}_{i+\frac{1}{2},j} = \bar{\kappa}_{\|} \bar{b}_x \left(\bar{b}_x \frac{\partial \bar{T}}{\partial x} + \bar{b}_y \frac{\partial \bar{T}}{\partial y} \right)$$

<u>Trivial to expand to 3D</u>

6 fluxes at cell interfaces.

Each flux obtained from 4 flux corners.

Each flux corner uses 8 neighboring cells.

$$\frac{\partial \bar{T}}{\partial x} = \frac{T_{i+1,j+1}^{n+1} + T_{i+1,j}^{n+1} - T_{i,j+1}^{n+1} - T_{i,j}^{n+1}}{2\Delta x},$$

$$\frac{\partial \bar{T}}{\partial y} = \frac{T_{i+1,j+1}^{n+1} + T_{i,j+1}^{n+1} - T_{i+1,j}^{n+1} - T_{i,j}^{n+1}}{2\Delta x},$$

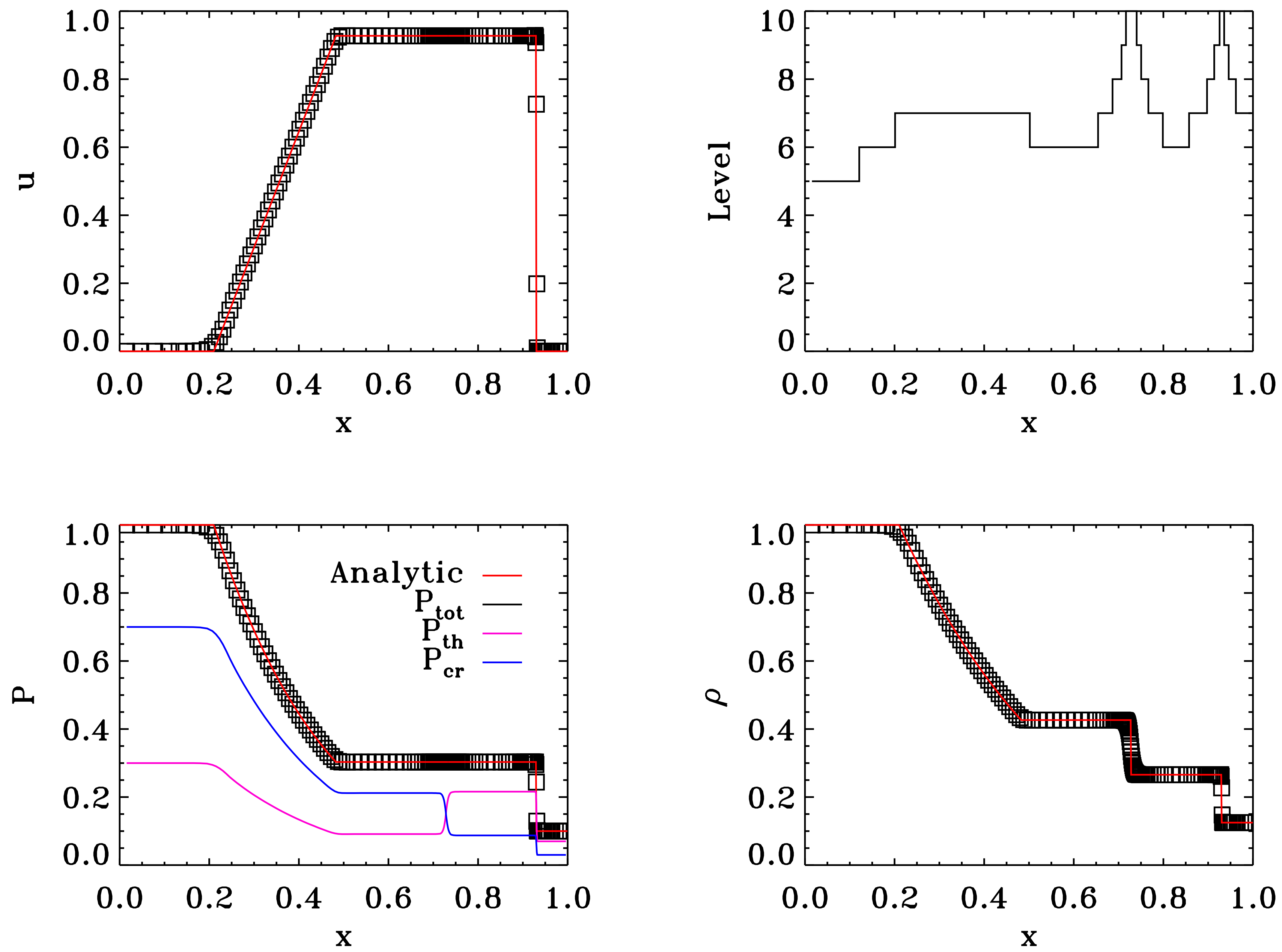
$$\bar{\kappa}_{\|} = \frac{\kappa_{\|i,j}^{n} + \kappa_{\|i+1,j}^{n} + \kappa_{\|i,j+1}^{n} + \kappa_{\|i+1,j+1}^{n}}{4}.$$

Centred symmetric scheme
(Günter et al, 2005)

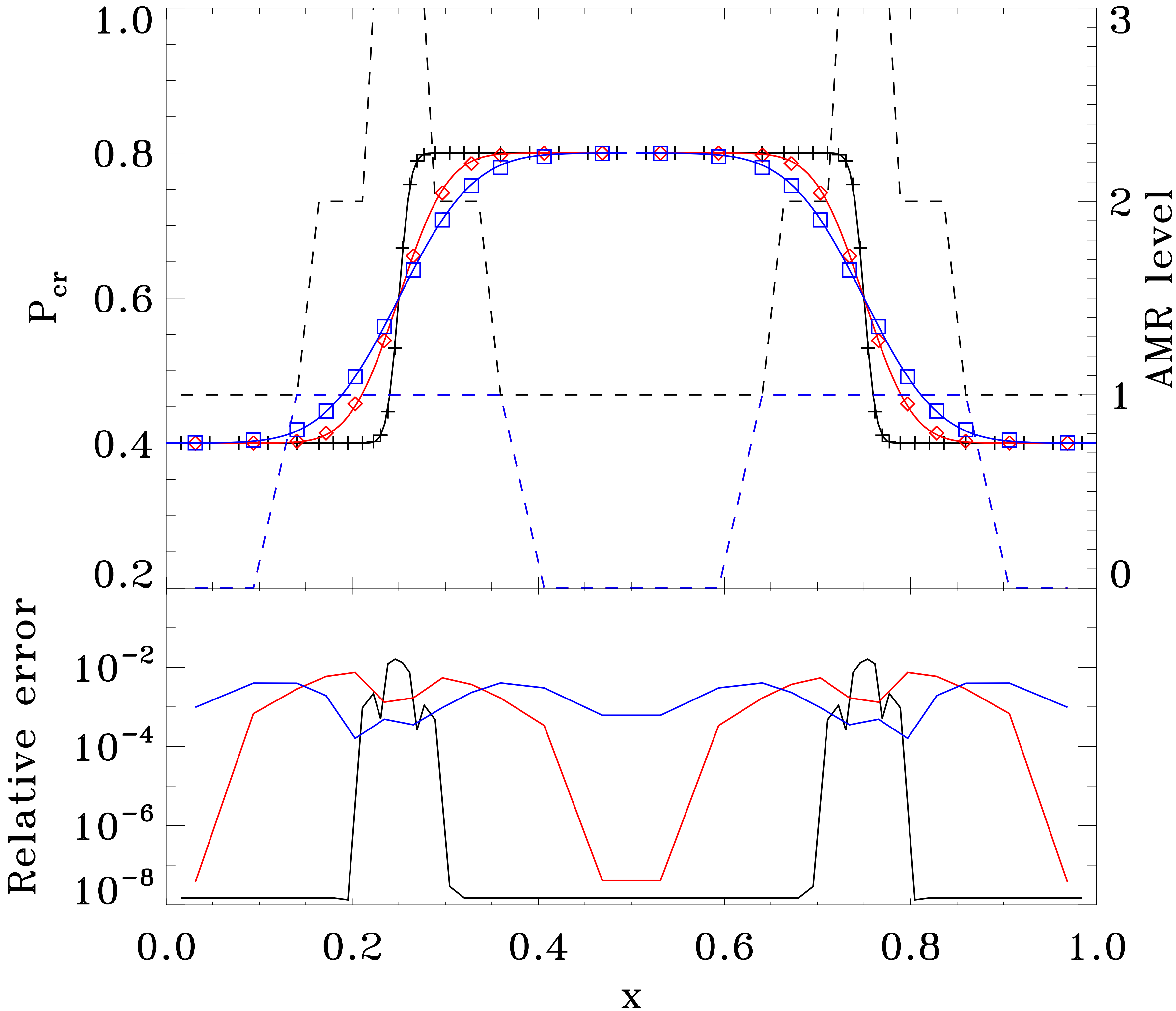
Limitations

- The solution to AC does not preserve monotonicity (i.e. negative temperatures)
  - ➡ Add a perpendicular diffusion coefficient and restore positive values after an AC step
  - ➡ (or use an asymmetric scheme w/ slope limiter, BUT need for bi-conjugate gradient method: HEAVIER)
- The Dirichlet boundary condition for AMR does not ensure energy conservation.
  - ➡ Live with it
  - ➡ (or use imposed fluxes at boundaries: more memory consumption and can create negative temperatures)
  - ➡ (or solve AC on the whole grid in one pass: more CPU expensive)

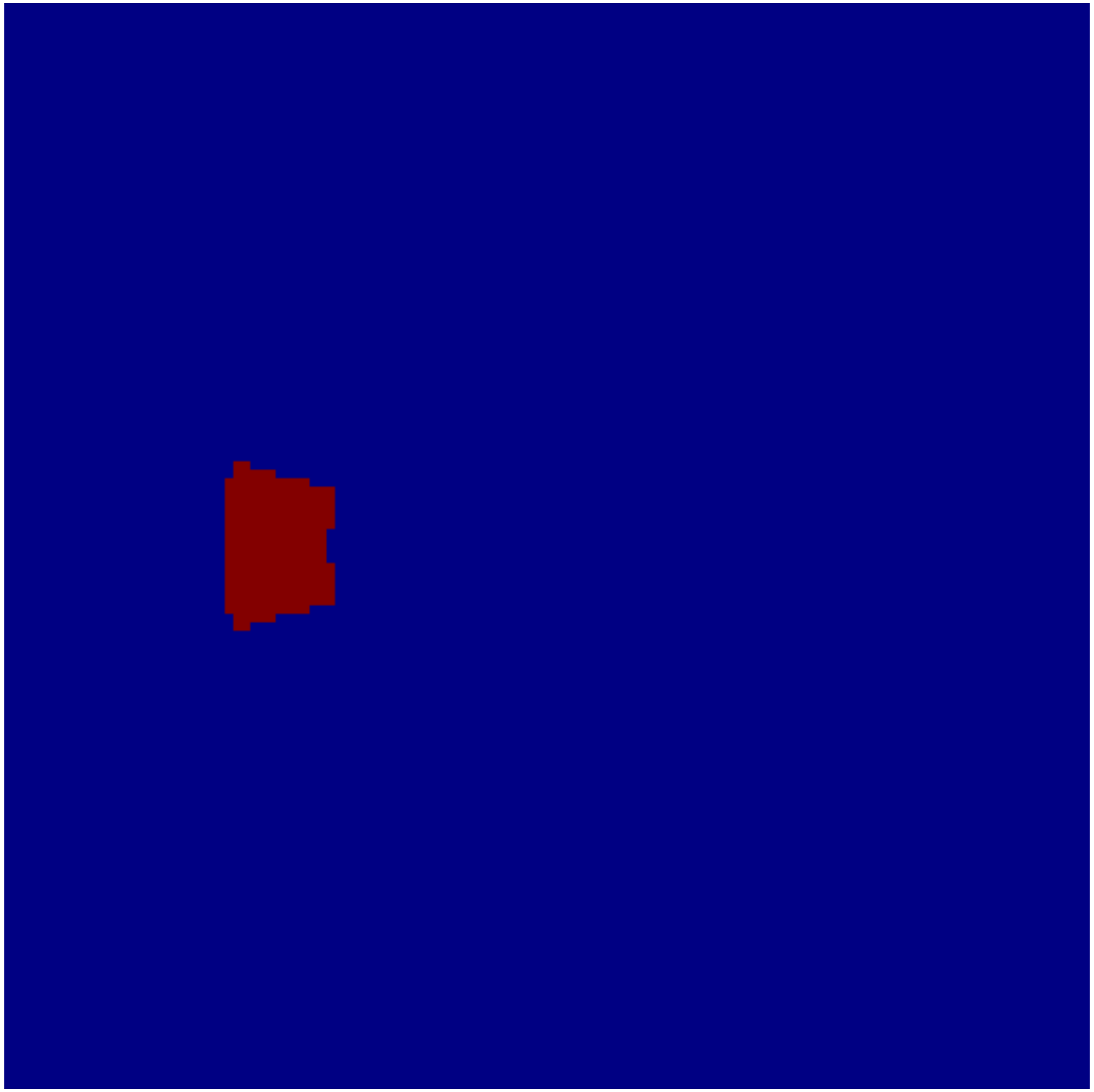
Test 1d: shock tube



Test 1d: diffusion of a step function



Test 2d: diffusion along circular B field



Energy diffusion along circular B field